COISOTROPIC SUBMANIFOLDS, LEAFWISE FIXED POINTS, AND PRESYMPLECTIC EMBEDDINGS

Fabian Ziltener (University of Toronto)

Let (M,ω) be a geometrically bounded symplectic manifold, $N \subseteq M$ a closed, regular (i.e. "fibering") coisotropic submanifold, and $\varphi : M \to M$ a Hamiltonian diffeomorphism. The main result of this article is that the number of leafwise fixed points of φ is bounded below by the sum of the $\mathbb{Z}_2$-Betti numbers of N, provided that the Hofer distance between φ and the identity is small enough and the pair (N,φ) is non-degenerate. The bound is optimal if there exists a $\mathbb{Z}_2$-perfect Morse function on N. A version of the Arnol'd-Givental conjecture for coisotropic submanifolds is also discussed. As an application, I prove a presymplectic non-embedding result.

Contents

1. Main results

Leafwise fixed points. Let (M,ω) be a symplectic manifold. We denote by $\mathrm{Ham}(M,\omega)$ the group of Hamiltonian diffeomorphisms (see Section 2). Let $\varphi \in \mathrm{Ham}(M,\omega)$ and $N \subseteq M$ be a coisotropic submanifold. We denote by $N_x := N_x^\omega \subseteq N$ the isotropic leaf through x (see Section 2). A leafwise fixed point of φ is by definition a point $x \in N$ such that $\varphi(x) \in N_x$. We denote by $\mathrm{Fix}(\varphi,N) := \mathrm{Fix}(\varphi,N,\omega)$ the set of such points. The first main result of this article addresses the following question:

Question A: Provided that φ is close to the identity in a suitable sense, what lower bound on the number $\big|\mathrm{Fix}(\varphi,N)\big|$ is there?

Note that if $N = M$ then $N_x = \{x\}$, for every $x \in N$, and hence $\mathrm{Fix}(\varphi, N)$ is the set $\mathrm{Fix}(\varphi)$ of ordinary fixed points of φ. In the other extreme case $\dim N = \dim M/2$ the submanifold N is Lagrangian, and we have $\mathrm{Fix}(\varphi, N) = N \cap \varphi^{-1}(N)$, provided that N is connected. In order to state the first main result, we denote by $A(M, \omega, N)$ the minimal area of (M, ω, N) (see (12) below), and by $d := d^{M,\omega}$ the Hofer distance (see (15)). We call N regular iff its isotropic leaf relation (see Section 2) is a closed subset and a submanifold of $N \times N$. Assuming that N is closed, this means that there exists a manifold structure on the set N_ω of isotropic leaves of N such that the canonical projection $\pi_N : N \to N_\omega$ is a smooth fiber bundle. For the definitions of (geometric) boundedness of (M, ω) and non-degeneracy for (N, φ) see Section 2. The former is a mild condition on (M, ω), examples include closed (compact without boundary) symplectic manifolds, cotangent bundles of closed manifolds, and symplectic vector spaces. Non-degeracy of (N, φ) naturally generalizes the usual non-degeneracy in the cases $N = M$ and $\dim N = \dim M/2$. For each topological space X, commutative ring R and integer i we denote by $b_i(X, R) := \mathrm{rank}_R\, H_i(X, R)$ the i-th Betti number of X with coefficients in R.

1. Theorem. *Let (M, ω) be a (geometrically) bounded symplectic manifold, and $N \subseteq M$ a closed, regular coisotropic submanifold. Then there exists a constant $C \in (0, \infty]$ such that $C \geq A(M, \omega, N)$ and the following holds. If $\varphi \in \mathrm{Ham}(M, \omega)$ is such that (N, φ) is non-degenerate and*

$$d(\varphi, \mathrm{id}) < C, \tag{1}$$

then

$$\big|\mathrm{Fix}(\varphi, N)\big| \geq \sum_{i=0}^{\dim N} b_i(N, \mathbb{Z}_2). \tag{2}$$

If $\mathrm{codim} N \neq 0, 1, \dim M/2$ then this theorem appears to be the first result implying that $\big|\mathrm{Fix}(\varphi, N)\big| \geq 2$, without assuming that φ is C^1-close to the identity. It generalizes a result for the case $\dim N = \dim M/2$, which is due to Yu. V. Chekanov, see the Main Theorem in [**Ch**]. The bound (2) is sharp, provided that there exists a $\mathbb{Z}_2$-perfect Morse function on N, see Theorem 2 below.

Examples. A large class of examples of regular coisotropic submanifolds is given as follows. Let (M, ω) be a symplectic manifold, and G a compact, connected Lie group with Lie algebra $\mathfrak{g}$. We fix a Hamiltonian action of G on M, and an (equivariant) moment map $\mu : M \to \mathfrak{g}^*$. Assume that μ is proper and the action of G on $N := \mu^{-1}(0) \subseteq M$ is free. Then N is a closed, regular coisotropic submanifold. As a concrete example, let $0 < k \leq n$ be integers, and consider $M := \mathbb{C}^{k \times n}$ with the standard symplectic structure $\omega := \omega_0$, and the action of the unitary group $G := \mathrm{U}(k)$

on $\mathbb{C}^{k\times n}$ by multiplication from the left. A moment map for this action is given by $\mu(\Theta) := \frac{i}{2}(\mathbf{1} - \Theta\Theta^*)$, and $N = \mu^{-1}(0)$ is the Stiefel manifold $V(k,n) := \big\{\Theta \in \mathbb{C}^{k\times n} \,\big|\, \Theta\Theta^* = \mathbf{1}\big\}$. The triple (M,ω,N) satisfies the hypotheses in Theorem 1. Furthermore, we have $A\big(\mathbb{C}^{k\times n},\omega_0,V(k,n)\big) = \pi$ (see Proposition 3 below), and $\sum_{i=0}^{\dim V(k,n)} b_i\big(V(k,n),\mathbb{Z}_2\big) = 2^k$. (This follows for example from [**GHV**], Theorem I on p. 224, and the fact that the integral cohomology of $V(k,n)$ is torsion-free.) Let $\varphi \in \mathrm{Ham}(\mathbb{C}^{k\times n},\omega_0)$ be such that $\big(V(k,n),\varphi\big)$ is non-degenerate and $d(\varphi,\mathrm{id}) < \pi$. Then Theorem 1 implies that $\big|\mathrm{Fix}(\varphi,V(k,n))\big| \geq 2^k$.

This bound is sharp, since there exists a $\mathbb{Z}_2$-perfect Morse function on $V(k,n)$ (see example (i) after Theorem 2 below). Moreover, the condition $C \geq A\big(\mathbb{C}^{k\times n},\omega_0,V(k,n)\big)$ in Theorem 1 is also sharp, in the sense that the conclusion of the theorem is wrong if we choose $C > A\big(\mathbb{C}^{k\times n},\omega_0,V(k,n)\big)$, see Proposition 3. Note that in the case $k=1$ we obtain an improvement of a result by H. Hofer, [**Ho**], Proposition 1.4. That result states that $\mathrm{Fix}(\varphi,S^{2n-1}) \neq \emptyset$, provided that $d_{\mathrm{c}}(\varphi,\mathrm{id}) \leq \pi$. Here d_{c} denotes the compactly supported Hofer distance (see (16) below).

Another family of examples of regular coisotropic submanifolds arises as follows. Let (X,σ) be a closed symplectic manifold, $\pi : E \to X$ a closed smooth fiber bundle, and $H \subseteq TE$ a horizontal subbundle. We define V^*E to be the vertical cotangent bundle of E. Its fiber over a point $e \in E$ is the space $T_e^*E_{\pi(e)}$. We denote the zero-section of this bundle by N. Furthermore, we define a closed two-form on V^*E as follows. We denote by $\pi_X : V^*E \to X$ the canonical projection, by ω_{can} the canonical symplectic form on T^*E, and by $\mathrm{pr}_e^H : T_eE \to T_eE_{\pi(e)}$ the linear projection along the subspace $H_e \subseteq T_eE$, for $e \in E$. We define

$$\iota_H : V^*E \to T^*E, \quad \iota_H(e,\alpha) := \big(e,\alpha\circ\mathrm{pr}_e^H\big), \quad \Omega_{\sigma,H} := \pi_X^*\sigma + \iota_H^*\omega_{\mathrm{can}}.$$

Then $\Omega_{\sigma,H}$ is a closed two-form on V^*E. Furthermore, by Proposition 3.2 in [**Ma**], there exists an open neighborhood M of the zero-section $N \subseteq V^*E$ on which $\Omega_{\sigma,H}$ is non-degenerate. We fix such an M. Then the submanifold $N \subseteq M$ is regular coisotropic (see Proposition 15 below). Assume now that the base manifold X is symplectically aspherical, i.e. $\int_{S^2} u^*\sigma = 0$, for every $u \in C^\infty(S^2,X)$. Then $A(M,\Omega_{\sigma,H},N) = \infty$ (see again Proposition 15 below). So in this case the only possible constant C as in Theorem 1 is ∞, and for this constant condition (1) is vacuous.

Idea of proof of Theorem 1. Assume that the hypotheses of Theorem 1 are satisfied. The strategy of the proof is to find a Lagrangian embedding of N into a suitable symplectic manifold, and then apply the Main Theorem in [**Ch**]. Recall that N_ω denotes the set of isotropic leaves of N. Since N is regular, there exists a unique manifold structure on N_ω such that the projection $\pi_N : N \to N_\omega$ is a submersion (see Lemma 24 below). We denote

by ω_N the unique symplectic structure on N_ω such that $\pi_N^*\omega_N = \omega$, and we define

$$\widetilde{M} := M \times N_\omega, \quad \widetilde{\omega} := \omega \oplus (-\omega_N), \tag{3}$$

$$\iota_N : N \to \widetilde{M}, \quad \iota_N(x) := (x, N_x), \quad \widetilde{N} := \iota_N(N). \tag{4}$$

Then ι_N is an embedding of N into $\widetilde{M}$ that is Lagrangian with respect to the symplectic form $\widetilde{\omega}$ on $\widetilde{M}$ (see Lemma 9 below). In order to satisfy the hypotheses of Chekanov's result, the inequality $A(M,\omega,N) \leq A(\widetilde{M},\widetilde{\omega},\widetilde{N})$ is crucial. It follows from Key Lemma 11 below. The idea of its proof is that given a smooth map $\widetilde{u} = (v, w') : \mathbb{D} \to \widetilde{M} = M \times N_\omega$ such that $\widetilde{u}(S^1) \subseteq \widetilde{N}$, we may lift w' to a map $w : [0,1] \times S^1 \to N$ and concatenate this with v. We thus obtain a map $u : \mathbb{D} \to M$ with boundary on an isotropic leaf, satisfying $\int u^*\omega = \int \widetilde{u}^*\widetilde{\omega}$. The method described here generalizes a standard way of reducing the case $\dim N = \dim M$ to the Lagrangian case, see for example [**Fl**].

Discussion of optimality. Let M be a manifold, $f : M \to \mathbb{R}$ a Morse function, and R a commutative ring. We denote by $\mathrm{Crit} f \subseteq M$ the set of critical points of f. Recall that f is called R-perfect iff $|\mathrm{Crit} f| = \sum_{i=0}^{dimM} b_i(M,R)$. The next result implies that the estimate (2) is sharp if there exists a $\mathbb{Z}_2$-perfect Morse function on the coisotropic submanifold N. It actually shows that in this case (2) is sharp, even if the condition (1) is replaced by the much stronger condition that φ is C^1-close to the identity. We denote by $\mathrm{Ham_c}(M,\omega)$ the group of compactly supported Hamiltonian diffeomorphisms of M.

2. Theorem. *Let (M,ω) be a symplectic manifold, $N \subseteq M$ a closed regular coisotropic submanifold, $f : N \to \mathbb{R}$ a Morse function, $\iota : M \to \mathbb{R}^{4n}$ an embedding, and $\varepsilon > 0$. Then there exists $\varphi \in \mathrm{Ham_c}(M,\omega)$ such that (N,φ) is non-degenerate, and*

$$\mathrm{Fix}(\varphi, N) = \mathrm{Crit} f, \quad \left\| \iota \circ \varphi \circ \iota^{-1} - \mathrm{id} \right\|_{C^1(\iota(M))} < \varepsilon.$$

The proof of this result relies on a normal form theorem for a neighborhood of N, which is due to Marle, and on the fact that fast almost periodic orbits of a vector field are constant. It also uses an estimate for the distance between the initial and the end point of a path x in foliation, assuming that these points lie in the same leaf, and that x is tangent to a given horizontal distribution.

Examples of manifolds admitting a $\mathbb{Z}_2$-perfect Morse function include the following:

(i) The real, complex and quaternonian Stiefel manifolds. (See [**TT**], the remarks after Ex. 3.14 on p. 197, and the definition of tautness on p. 182.)

(ii) Compact symmetric spaces that admit a symmetric embedding into Euclidian space. This includes the real, complex and quaternonian Grassmannian. (See [**DV**] Theorem 1.2 and example 1. on p. 7.)
(iii) Quotients G/T, where G is a compact connected semi-simple Lie group, and $T \subseteq G$ is a maximal torus. (This follows for example from [**Du**] Theorems 4 and 5 on p. 125.) Note that for $G := \mathrm{SU}(n)$ G/T is diffeomorphic to the manifold of complete flags in $\mathbb{C}^n$.
(iv) Symplectic manifolds that admit a Hamiltonian S^1-action whose fixed points are isolated, see for example [**GGK**], the theorem on p. 22.
(v) Simply connected closed manifolds of dimension at least 6, whose homology with $\mathbb{Z}$-coefficients is torsion-free. (This follows from [**An**] Theorem 4.2.4(ii) on p. 112, Definition 4.1.1 on p. 106, and formula (4.2.4) on p. 111.)

Note that example (iv) generalizes example (iii). Observe also that the product of two manifolds allowing a $\mathbb{Z}_2$-perfect Morse function, has the same property. (This follows from [**An**] Theorem 4.1.5 on p. 109 and the Künneth formula.)

Consider now $\mathbb{C}^{k\times n}$ with the standard symplectic form ω_0, and the Stiefel manifold $V(k,n) \subseteq \mathbb{C}^{k\times n}$. Then by the next result the condition $C \geq A\big(\mathbb{C}^{k\times n},\omega_0,V(k,n)\big)$ in Theorem 1 is sharp. We denote by d_{c} the compactly supported Hofer distance (see Section 2).

3. Proposition. *We have* $A\big(\mathbb{C}^{k\times n},\omega_0,V(k,n)\big) = \pi$*. Furthermore, for every* $C > \pi$ *there exists* $\varphi \in \mathrm{Ham}_{\mathrm{c}}(\mathbb{C}^{k\times n},\omega_0)$ *such that* $d_{\mathrm{c}}(\varphi,\mathrm{id}) < C$ *and* $\varphi(V(k,n)) \cap V(k,n) = \emptyset$*.*

Arnol'd-Givental conjecture (AGC) for coisotropic submanifolds. Recall that a map from a set to itself is called an involution iff applying it twice yields the identity. Furthermore, a diffeomorphism ψ from a symplectic manifold (M,ω) to itself is called anti-symplectic iff $\psi^*\omega = -\omega$. The following conjecture naturally generalizes the usual (Lagrangian) AGC to "product"-coisotropic submanifolds in products of symplectic manifolds.

Conjecture. *Let* (M_i,ω_i)*,* $i = 1,2$ *be symplectic manifolds, with* M_1 *closed, and let* $L \subseteq M_2$ *be a closed Lagrangian submanifold. Consider the product* $M := M_1 \times M_2$ *with the symplectic structure* $\omega := \omega_1 \oplus \omega_2$*, and let* $N := M_1 \times L$*. Assume that there exists an anti-symplectic involution* $\psi : M_2 \to M_2$ *such that* $\mathrm{Fix}(\psi) = L$*. Let* $\varphi \in \mathrm{Ham}(M,\omega)$ *be such that the pair* (N,φ) *is non-degenerate. Then inequality (2) holds.*

In the case in which M_1 is a point this is the usual (Lagrangian) AGC. (See for example [**Fr**], where it is assumed that M is compact.)

4. Proposition. *If the* Lagrangian *AGC is true then the same holds for the above Conjecture.*

An application. By definition a presymplectic manifold is a pair (M,ω), where M is a manifold, and ω is a closed two-form on M of constant corank $\mathrm{corank}\,\omega$ (see Section 2). We say that a presymplectic manifold (M',ω') embeds into a presymplectic manifold (M,ω) iff there exists an embedding $\psi : M' \to M$ such that $\psi^*\omega = \omega'$. The following question generalizes the symplectic and Lagrangian embedding problems.

Question B: Given two presymplectic manifolds, does one of them embed into the other one?

Note that in the case $\dim M' + \mathrm{corank}\,\omega' > \dim M + \mathrm{corank}\,\omega$ there does not even exist any immersion $\psi : M' \to M$ satisfying $\psi^*\omega = \omega'$. (This follows from Proposition 12 below.) The next result is concerned with the "critical case" in which ">" is replaced by "=" above. It is a consequence of Theorem 1. A presymplectic manifold (M,ω) is called regular iff its isotropic leaf relation is a closed subset of $M \times M$ and a submanifold.

5. Corollary. *Let (M,ω) be a bounded and aspherical symplectic manifold, and (M',ω') a closed, regular presymplectic manifold of corank $\dim M - \dim M'$. Assume that every compact subset of M can be displaced in a Hamiltonian way, and that M' has a simply-connected isotropic leaf. Then (M',ω') does not embed into (M,ω).*

Examples. As an example, let (X,σ) and (X',σ') be symplectic manifolds, the former bounded and aspherical and the latter closed. Let F be a closed simply-connected manifold. Assume that $\dim X + 2 = \dim X' + 2\dim F$. Then the hypotheses of Corollary 5 are satisfied with

$$M := X \times \mathbb{R}^2, \quad \omega := \sigma \oplus \omega_0, \quad M' := X' \times F, \quad \omega' := \sigma' \oplus 0.$$

As a more specific example, let (X',σ') be a closed aspherical symplectic manifold, and $k \geq 2$ and $0 \leq \ell \leq k$ be integers. We define

$$(M,\omega) := \big(X' \times \mathbb{R}^{2(k-\ell)} \times \mathbb{R}^\ell, \sigma' \oplus \omega_0 \oplus 0\big), \quad (M',\omega') := \big(X' \times S^k, \sigma' \oplus 0\big).$$

Then (M',ω') does not embed into (M,ω). To see this, observe that every embedding of (M',ω') into (M,ω) gives rise to an embedding of (M',ω') into $\big(X' \times \mathbb{R}^{2k}, \sigma' \oplus \omega_0\big)$, by composition with the canonical inclusion $M \to X' \times \mathbb{R}^{2k}$. Hence the statement follows from Corollary 5.

However, in this example there exists an embedding $\psi : M' \to M$ such that $\psi^*[\omega] = [\omega']$, provided that $\ell < k$. We may for example choose any embedding $\iota : S^k \to \mathbb{R}^{2(k-\ell)} \times \mathbb{R}^\ell$ and define $\psi := \mathrm{id}_{X'} \times \iota$. Furthermore, if $\ell = 0$ then there exists an *immersion* $\psi : M' \to M$ satisfying $\psi^*\omega = \omega'$. To see this, note that the Whitney map

$$f : S^k \subseteq \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^{2k} \cong \mathbb{C}^k, \quad f(a,x) := (1+ai)x$$

is a Lagrangian immersion. (See [**ACL**], Example I.4.3, p. 17.) The map $\psi := \mathrm{id}_{X'} \times f$ has the desired properties.

Further research. A further direction of research is to replace the closeness assumption (1) by a suitable monotonicity assumption. This requires a definition of a Maslov map of the triple (M, ω, N). In a forthcoming article [**Zi1**] I give such a definition.

Related results. In the extreme cases $N = M$ and $\dim N = \dim M/2$ Question A has been investigated a lot. For some references, see for example [**MS**], Sec. 9.1., p. 277, and [**Gin**], Sec. 1.1. p.112. If (M, ω) is a closed symplectic manifold, and $\varphi \in \mathrm{Ham}(M, \omega)$ is such that every $x \in \mathrm{Fix}(\varphi)$ is non-degenerate then Arnol'd [**Ar**] conjectured that $|\mathrm{Fix}(\varphi)| \geq |\mathrm{Crit} f|$ for every Morse function $f : M \to \mathbb{R}$.

The general coisotropic case was first considered by J. Moser. He proved that $\big|\mathrm{Fix}(\varphi, N)\big| \geq 2$ if M is simply connected, ω is exact, and the C^1-distance $d^{C^1}(\varphi, \mathrm{id})$ is sufficiently small, see the theorem on p. 19 in [**Mos**]. (In fact, he showed that $\big|\mathrm{Fix}(\varphi, N)\big|$ is bounded below by the Lusternik-Schnirelmann category of N, see Proposition 5, p.31 in [**Mos**].) A. Banyaga [**Ba**] removed the simply connectedness and exactness conditions. Because of the C^1-closeness condition these are local results. Global results were first obtained by I. Ekeland and H. Hofer [**EH**, **Ho**]. For N a closed connected hypersurface in $\mathbb{R}^{2n}$ of restricted contact type they gave several criteria under which $\mathrm{Fix}(\varphi, N) \neq \emptyset$, allowing for interesting cases in which $d^{C^1}(\varphi, \mathrm{id})$ is big. For example, in Theorem 1.6 in [**Ho**] it is assumed that the compactly supported Hofer distance $d_{\mathrm{c}}(\varphi, \mathrm{id})$ is bounded above by the Ekeland-Hofer capacity $c_{EH}(N)$. Recall here that a coisotropic submanifold $N \subseteq M$ of codimension k is said to be of contact type iff there exist one-forms $\alpha_1, \ldots, \alpha_k$ on N such that $d\alpha_i = \omega$, for $i = 1, \ldots, k$, and $\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \omega|_N^{n-k}$ does not vanish anywhere on N. Here $\omega|_N$ denotes the pullback of ω under the inclusion of N into M. N is said to be of restricted contact type iff the α_i's extend to global primitives of ω. D. Dragnev ([**Dr**], Theorem 1.3) proved a similar result for general codimension of N, replacing $c_{EH}(N)$ by the Floer Hofer capacity of N, and assuming that N is only of contact type.

Generalizing in another direction, V. Ginzburg proved a version of Hofer's result for subcritical Stein manifolds, replacing c_{EH} by some homological capacity c_{hom} (see [**Gin**], Theorem 2.9 p. 122). This result in turn was recently extended by B. Gürel [**Gü**] to the coisotropic case (with c_{hom} replaced by some constant depending on N). For general codimension of N, Ginzburg observed that $\mathrm{Fix}(\varphi, N) \neq \emptyset$ if the isotropic foliation of N is a fibration (i.e. N is regular) and "φ is not far from id in a suitable sense", see [**Gin**], Example 1.3 p. 113. His argument is based on the fact that in this case the leaf relation is a Lagrangian submanifold of the product $M \times M$, equipped with the symplectic form $\omega \oplus (-\omega)$. Lately, P. Albers and U. Frauenfelder

proved that $\mathrm{Fix}(\varphi, N) \neq \emptyset$, if (M,ω) is convex at infinity, $N \subseteq M$ is a closed hypersurface of restricted contact type, and $d_{\mathrm{c}}(\varphi, \mathrm{id}) < A(M,\omega,N)$. If in addition the Rabinowitz action functional of the Hamiltonian function generating φ is Morse, then they showed that $\big|\mathrm{Fix}(\varphi,N)\big| \geq \sum_i b_i(N,\mathbb{Z}_2)$. (See Theorems A and B in [**AF**].) A problem related to Question A is to find a lower bound on the displacement energy of a coisotropic submanifold. Recent work on this problem other than the one already mentioned has been carried out by E. Kerman in [**Ke**].

Note that regularity of N and the contact type condition do not imply each other. For example, every Lagrangian submanifold is regular. However, if N is a closed connected Lagrangian submanifold of contact type then it is a torus, see for example [**Gin**], Example 2.2 (iv), p. 118. On the other hand, consider $(M,\omega) := (\mathbb{C}^2, \omega_0)$, fix an irrational number $a > 0$, and define $H : \mathbb{C}^2 \to \mathbb{R}$ by $H(z,w) := |z|^2 + |w|^2/a$. Then the ellipsoid $N := H^{-1}(1) \subseteq M$ is a hypersurface of restricted contact type, since the region bounded by N is convex. However, the only compact isotropic leaves are the circles $\big\{(z,0)\,\big|\,|z|^2 = 1\big\}$ and $\big\{(0,w)\,\big|\,|w|^2 = a\big\}$. (The leaves are the integral curves of the Hamiltonian vector field of H.) Hence N is not regular. Note that "restricted contact type" is a global condition on (M,ω,N), whereas regularity is a condition only on $(N, \omega|_N)$.

If N is of restricted contact type then it is stable (see Definition 2.1 p. 117 in [**Gin**]). Regularity and stability can be seen as "dual" conditions in the following sense. Namely, $(N, \omega|_N)$ is regular if and only if it fibers into isotropic submanifolds, whereas it is stable if and only if some neighborhood of N fibers as a family of coisotropic submanifolds containing N, see [**Gin**] Proposition 2.6, p. 120. Observe also that V. Ginzburg constructed a closed hypersurface $N \subseteq \mathbb{R}^{2n}$ without any closed characteristic, see [**Gin**], Example 7.2 p. 158. This means that $A(\mathbb{R}^{2n}, \omega_0, N) = \infty$. Furthermore, B. Gürel gave an example of a hypersurface $N \subseteq \mathbb{R}^4$ such that $A(\mathbb{R}^4, \omega_0, N) = \infty$, and for every $\varepsilon > 0$ there exists $\varphi \in \mathrm{Ham}(M,\omega)$ satisfying $\mathrm{Fix}(\varphi, N) = \emptyset$ and $d_{\mathrm{c}}(\varphi, \mathrm{id}) < \varepsilon$ (see [**Gü**]). This shows that one may not completely drop the regularity or stability condition on N if one wants to prove existence of leafwise fixed points.

Let now M, ω, M' and ω' be as in the hypothesis of Corollary 5. Assume that (M,ω) is the product of some bounded symplectic manifold with $(\mathbb{R}^2, \omega_0)$ and that M' is simply connected. Then the statement of the corollary follows from the comments after Example 2.2.8. on pp. 288 and 289 in [**ALP**], using Proposition 12 and Lemma 24 below. Like the proof of Corollary 5, that argument is based on the fact that the image of the map ι_N defined in (4) is a Lagrangian submanifold of $M \times N_\omega$. However, since it does not involve the Key Lemma 11, the assumption that M' is simply connected is needed there. On the other hand, if ω is exact then Corollary 5 can be deduced from Example 1.7, p.115 in [**Gin**], using again Proposition

12 and Lemma 24. Furthermore, if the presymplectic manifold (M',ω') is stable then a similar non-embedding result can be deduced from Theorem 2.7 (ii) p. 121 in [**Gin**].

Organization of the article. Section 2 contains some background on foliations, presymplectic manifolds, coisotropic submanifolds, leafwise fixed points, and Hamiltonian diffeomorphisms. In this section, the linear holonomy of a foliation along a path in a leaf, and based on this, non-degeneracy of a pair (N,φ), are defined. In Section 3 Chekanov's theorem is restated (Theorem 7), and the relevant properties of the map ι_N and the subset $\widetilde{N}\subseteq\widetilde{M}$ (as in (4)) are established (Lemmas 9 and 11). Based on this, the main results are proven in Section 4. Appendix A.1 contains some background about presymplectic geometry, on the embedding of a smooth fiber bundle over a symplectic base into its vertical cotangent bundle, and three other elementary results from symplectic geometry. In Appendix A.2 the result is proven that is used in the definition of linear holonomy. Furthermore, an estimate for a tangent path of a horizontal distribution in a foliation is proven. Finally, Appendix A.3 contains results about smooth structures on the quotient set of an equivalence relation, fast almost periodic orbits of a vector field, and a measure theoretic lemma.

Acknowledgments. I am specially indebted to Yael Karshon for numerous enlightening discussions and her continuous support and encouragement. She also made me aware that the conclusion of Corollary 5 follows from an easy cohomological argument if $(M,\omega):=(\mathbb{R}^{2n},\omega_0)$ and $(M',\omega'):=\big(X'\times F,\sigma'\oplus 0\big)$, with X' closed, $\dim X'>0$ and σ' symplectic. I would also like to thank Viktor Ginzburg, Chris Woodward and Masrour Zoghi for interesting conversations, and Urs Frauenfelder for useful hints. It was Chris Woodward from whom I learned about the construction of the Lagrangian submanifold $\widetilde{N}\subseteq\widetilde{M}$ in the case of a Hamiltonian Lie group action, with $N:=\mu^{-1}(0)$.

2. Background

Notation, manifolds. We denote by $\mathbb{N}$ the positive integers, by $\mathbb{D},S^1\subseteq\mathbb{R}^2$ the closed unit disk and the unit circle, and for $r>0$ by $B_r\subseteq\mathbb{R}^2$ the open ball of radius r. For two vector spaces V and V' and a linear map $\Psi:V'\to V$ we denote by $\ker\Psi$ and $\mathrm{im}\Psi$ its kernel and image, and by $\Psi^*:V^*\to V'^*$ its adjoint map. Let M be a set. By a *smooth structure* on M we mean a maximal smooth (C^∞) atlas $\mathcal{A}$ of charts $\varphi:U\subseteq M\to\mathbb{R}^n$. (Hence M does not have any boundary.) Assume that M is equipped with a smooth structure. We denote by $C^\infty(M,\mathbb{R})$ and $C^\infty_{\mathrm{c}}(M,\mathbb{R})$ the set of smooth and compactly supported smooth functions, respectively. We call $(M,\mathcal{A})$ a *manifold* iff the topology on M induced by $\mathcal{A}$ is Hausdorff and

second countable. *Submanifolds* of M are by definition embedded. For a smooth time-dependent vector field X on a manifold M and $t \in \mathbb{R}$ we denote by $\varphi_X^t : M \to M$ its time-t-flow (if it exists).

Foliations, regularity, and linear holonomy. We recollect some basic definitions and facts about foliations. For more details, see for example the book [**MM**]. The definition of linear holonomy given below will be needed to define non-degeneracy of a pair (N, φ) as in section 1. I am not aware of a reference in which the linear holonomy is defined in precisely this way. However, the basic idea of its definition is standard, see for example [**MM**].

Let M be a set, $0 \leq k \leq n$ integers, $U, U' \subseteq M$ subsets, and $\varphi : U \to \mathbb{R}^n$, $\varphi' : U' \to \mathbb{R}^n$ injective maps. We denote by $\pi_1 : \mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}^{n-k}$ the canonical projection onto the first factor. We call (U, φ) and (U', φ') (n, k)-compatible iff $\varphi(U \cap U') \subseteq \mathbb{R}^n$ is open, $\varphi' \circ \varphi^{-1} : \varphi(U \cap U') \to \varphi'(U \cap U')$ is a diffeomorphism, and for every $\xi \in \mathbb{R}^{n-k}$ the map

$$\pi_1 \circ \varphi' \circ \varphi^{-1}(\xi, \cdot) : \big\{ \eta \in \mathbb{R}^k \,\big|\, (\xi, \eta) \in \varphi(U \cap U') \big\} \to \mathbb{R}^{n-k}$$

is locally constant. We define an (n, k)-atlas on M to be a set $\mathcal{A}$ of pairs (U, φ) as above, such that $\bigcup_{(U,\varphi) \in \mathcal{A}} U = M$ and each two pairs in $\mathcal{A}$ are (n, k)-compatible. An (n, k)-foliation on M is defined to be a maximal (with respect to inclusion) (n, k)-atlas on M. Let $\mathcal{F}$ be an (n, k)-foliation on M. We endow M with the smooth structure induced by $\mathcal{F}$, and for $x \in M$, we define

$$T_x\mathcal{F} := d\varphi(x)^{-1}\big(\{0\} \times \mathbb{R}^k\big) \subseteq T_xM,$$

where $(U, \varphi) \in \mathcal{F}$ is a chart such that $x \in U$. We define the leaf through a point $x_0 \in M$ to be the set

$$\mathcal{F}_{x_0} := \big\{ x(1) \,\big|\, x \in C^\infty([0,1], M) : x(0) = x_0,\ \dot{x}(t) \in T_{x(t)}\mathcal{F},\ \forall t \big\} \subseteq M.$$

The leaf relation is defined to be the set

$$R^{\mathcal{F}} := \big\{ (x_0, x_1) \in M \times M \,\big|\, x_1 \in \mathcal{F}_{x_0} \big\}.$$

It is an equivalence relation on M. The collection of the subspaces $T_x\mathcal{F}$, with $x \in M$, is an involutive distribution $T\mathcal{F}$ on M, called the tangent bundle to $\mathcal{F}$. We denote by $N\mathcal{F} := TM/T\mathcal{F}$ the normal bundle, and by $\mathrm{pr}^{\mathcal{F}} : TM \to N\mathcal{F}$ the canonical projection, and for $x \in M$, we write $N_x\mathcal{F} := (N\mathcal{F})_x$.

Let now $(M, \mathcal{A})$ be a manifold. By a foliation on $(M, \mathcal{A})$ we mean a foliation $\mathcal{F}$ on M that induces $\mathcal{A}$. In this case we call the pair $(M, \mathcal{F})$ a foliated manifold. Note that if E is an involutive distribution on M then by Frobenius' Theorem there exists a unique foliation $\mathcal{F}^E$ on M such that $E_x = d\varphi(x)^{-1}\big(\{0\} \times \mathbb{R}^k\big)$ for every chart $(U, \varphi) \in \mathcal{F}^E$ for which $x \in U$. We call a foliated manifold $(M, \mathcal{F})$ *regular* iff $R^{\mathcal{F}}$ is a closed subset and submanifold of $M \times M$. By Lemma 24 below this holds if and only if there exists a manifold structure on the quotient $M/R^{\mathcal{F}}$ such that the canonical projection from M to $M/R^{\mathcal{F}}$ is a submersion. Furthermore, such a structure

is unique. If $\mathcal{F}$ is regular then the leaves are closed subsets and submanifolds of M. If $(M,\mathcal{F})$ is a foliated manifold and $H\subseteq TM$ a distribution then we call H *($\mathcal{F}$-)horizontal* iff for every $x\in M$ we have $T_xM=H_x\oplus T_x\mathcal{F}$.

Let $(M,\mathcal{F})$ be a foliated manifold, F a leaf of $\mathcal{F}$, $a\leq b$, and $x\in C^\infty([a,b],F)$ a path. The *linear holonomy of $\mathcal{F}$ along x* is a linear map $\mathrm{hol}^{\mathcal{F}}_x:N_{x(a)}\mathcal{F}\to N_{x(b)}\mathcal{F}$. Its definition is based on the following result.

6. Proposition. *Let $M,\mathcal{F},F,a,b$ and x be as above, N a manifold, and $y_0\in N$. Then the following statements hold.*

(a) For every linear map $T:T_{y_0}N\to T_{x(a)}M$ there exists a map $u\in C^\infty([a,b]\times N,M)$ such that

$$u(\cdot,y_0)=x,\quad \mathcal{F}_{u(t,y)}=\mathcal{F}_{u(a,y)},\,\forall t\in[a,b],\,y\in N, \tag{5}$$

$$d(u(a,\cdot))(y_0)=T. \tag{6}$$

(b) Let $u,u'\in C^\infty([a,b]\times N,M)$ be maps satisfying (5), such that

$$\mathrm{pr}^{\mathcal{F}}d(u(a,\cdot))(y_0)=\mathrm{pr}^{\mathcal{F}}d(u'(a,\cdot))(y_0). \tag{7}$$

Then $\mathrm{pr}^{\mathcal{F}}d(u(b,\cdot))(y_0)=\mathrm{pr}^{\mathcal{F}}d(u'(b,\cdot))(y_0)$.

We choose a linear map $T:N_{x(a)}\mathcal{F}\to T_{x(a)}M$, such that $\mathrm{pr}^{\mathcal{F}}T=\mathrm{id}_{N_{x(a)}\mathcal{F}}$. Furthermore, we define $N:=N_{x(a)}\mathcal{F}$ and $y_0:=0$, and we choose a map $u\in C^\infty\big([a,b]\times N_{x(a)}\mathcal{F},M\big)$ such that (5) and (6) hold. Here in (6) we canonically identified $T_0\big(N_{x(a)}\mathcal{F}\big)=N_{x(a)}\mathcal{F}$. We define

$$\mathrm{hol}^{\mathcal{F}}_x:=\mathrm{pr}^{\mathcal{F}}d(u(b,\cdot))(0):N_{x(a)}\mathcal{F}(=T_0(N_{x(a)}\mathcal{F}))\to N_{x(b)}\mathcal{F}. \tag{8}$$

It follows from Proposition 6 that this map is well-defined. It can be viewed as the linearization of the holonomy of a foliation as defined for example in Sec. 2.1 in the book [**MM**].

Presymplectic manifolds and symplectic quotients. By a presymplectic vector space we mean a real vector space V together with a skew-symmetric bilinear form ω. Let (V,ω) be such a pair. For every linear subspace $W\subseteq V$ we denote by $W^\omega:=\big\{v\in V\,\big|\,\omega(v,w)=0,\,\forall w\in W\big\}$ its ω-complement. The subspace W is called coisotropic iff $W^\omega\subseteq W$. We define $\mathrm{corank}\,\omega:=\dim V^\omega$. A presymplectic structure on a manifold M is a closed two-form ω on M, such that $\mathrm{corank}\,\omega_x$ does not depend on $x\in M$. This number is called the corank of ω. Let (M,ω) be a presymplectic manifold. The *isotropic distribution* $TM^\omega=\big\{(x,v)\,\big|\,x\in M,\,v\in T_xM^\omega\big\}\subseteq TM$ is involutive. (This follows for example as in the proof of Lemma 5.33 in the book [**MS**].) We call $\mathcal{F}^{TM^\omega}$ the isotropic (or characteristic) foliation on M. We call (M,ω) *regular* iff $\mathcal{F}^{TM^\omega}$ is regular. Assume that (M,ω) is regular. We denote by M_ω the set of isotropic leaves of M, by $\pi_{M,\omega}:M\to M_\omega$ the canonical projection, and by $\mathcal{A}_{M,\omega}$ the unique manifold structure on M_ω such that $\pi_{M,\omega}$ is a submersion. There exists a unique symplectic form

ω_M on M_ω satisfying $\pi^*_{M,\omega}\omega_M = \omega$. The triple $\big(M_\omega, \mathcal{A}_{M,\omega}, \omega_M\big)$ is called the *symplectic quotient* of (M,ω). If M is also closed then by a result by C. Ehresmann the quadruple $\big(M, M_\omega, \mathcal{A}_{M,\omega}, \pi_{M,\omega}\big)$ is a smooth (locally trivial) fiber bundle, see the proposition on p. 31 in [**Eh**].

On the other hand, given a smooth fiber bundle (E,B,π) and a symplectic form σ on B, the pair $(E,\pi^*\sigma)$ is a regular presymplectic manifold. Let (M,ω) be a presymplectic manifold, and $H \subseteq TM$ a distribution. We call H *(ω-)horizontal* iff it is $\mathcal{F}^{TM^\omega}$-horizontal. This means that for every $x \in M$ we have $T_xM = H_x \oplus T_xM^\omega$. Assume that H is horizontal. This gives rise to a closed two-form $\Omega_{\omega,H}$ on the manifold $(TM^\omega)^*$, as follows. For $x \in M$ we denote by $\mathrm{pr}^H_x : T_xM \to T_xM^\omega$ the linear projection along the subspace $H_x \subseteq T_xM$, and we define

$$\iota_H : (TM^\omega)^* \to T^*M, \quad \iota_H(x,\alpha) := \big(x, \alpha \circ \mathrm{pr}^H_x\big). \tag{9}$$

We denote by $\pi : (TM^\omega)^* \to M$ the canonical projection, and by ω_{can} the canonical symplectic form on T^*M. We define

$$\Omega_{\omega,H} := \pi^*\omega + \iota_H^*\omega_{\mathrm{can}}. \tag{10}$$

By a result by C.-M. Marle there exists an open neighborhood of the zero section on which $\Omega_{\omega,H}$ is non-degenerate, see Proposition 3.2 in [**Ma**].

Coisotropic submanifolds and leafwise fixed points. Let (M,ω) be a presymplectic manifold. A submanifold $N \subseteq M$ is called coisotropic iff for every $x \in N$ the subspace $T_xN \subseteq T_xM$ is coisotropic. This holds if and only if the restriction $\omega|_N$ of ω to N (i.e. the pull-back under the inclusion map) is a presymplectic form satisfying $\dim N + \operatorname{corank}\omega|_N = \dim M + \operatorname{corank}\omega$. (This follows from Proposition 12 below.) If N is coisotropic then $2\dim N \geq \dim M + \operatorname{corank}\omega$. In the extreme case $\operatorname{corank}\omega = 0$ (i.e. ω symplectic) and $\dim N = \dim M/2$ the submanifold N is called Lagrangian. As an example, let F be a manifold, and (X,σ) a symplectic manifold. We denote by ω_{can} the canonical two-form on T^*F, and define $\omega := \sigma \oplus \omega_{\mathrm{can}}$. Then $X \times F$ is a coisotropic submanifold of $(X \times T^*F, \omega)$. As another example, every hypersurface (i.e. real codimension one submanifold) of a symplectic manifold is coisotropic.

Let $N \subseteq M$ be a coisotropic submanifold. For a point $x \in N$ we denote by $N_x := N^\omega_x \subseteq N$ the isotropic leaf through x. Furthermore, we denote by N_ω the set of all isotropic leaves of N, and by $\pi_N : N \to N_\omega$ the canonical projection. We define the *action spectrum* and the *minimal area* of (M,ω,N) as

$$S(M,\omega,N) := \left\{ \int_{\mathbb{D}} u^*\omega \,\middle|\, u \in C^\infty(\mathbb{D},M) : \exists F \in N_\omega : u(S^1) \subseteq F \right\}, \tag{11}$$

$$A(M,\omega,N) := \inf\big(S(M,\omega,N) \cap (0,\infty)\big) \in [0,\infty]. \tag{12}$$

Furthermore, we denote the linear holonomy $\mathrm{hol}^{\mathcal{F}^{TN^\omega}}$ by $\mathrm{hol}^{\omega,N}$. We call N *regular* iff the presymplectic manifold $(N,\omega|_N)$ is regular (i.e. the foliation $\mathcal{F}^{TN^\omega}$ on N is regular). Note that such an N is sometimes called "fibering".

Let $\varphi : M \to M$ be a map. We say that a point $x \in N$ is leafwise fixed under φ iff $\varphi(x) \in N_x$. We denote by $\mathrm{Fix}(\varphi,N) = \mathrm{Fix}(\varphi,N,\omega)$ the set of such points. Assume now that $\operatorname{corank}\omega = 0$, i.e. that ω is symplectic, and that φ is smooth. We call (N,φ,ω) (or simply (N,φ)) *non-degenerate* iff the following holds. For $x_0 \in N$ we denote by $\mathrm{pr}_{x_0} : T_{x_0}N \to T_{x_0}N/(T_{x_0}N)^\omega$ the canonical projection. Let $F \subseteq N$ be an isotropic leaf, and $x \in C^\infty([0,1],F)$ a path. Assume that $\varphi(x(0)) = x(1)$, and let $v \in T_{x(0)}N \cap T_{x(0)}\varphi^{-1}(N)$ be a vector. Then $v \neq 0$ implies that

$$\mathrm{hol}_x^{\omega,N}\mathrm{pr}_{x(0)}v \neq \mathrm{pr}_{x(1)}d\varphi(x(0))v. \tag{13}$$

Note that in the case $N = M$ this condition means that for every fixed point x_0 of φ the differential $d\varphi(x_0)$ does not have 1 as an eigenvalue. Furthermore, in the case that N is Lagrangian the condition means that every connected component $N' \subseteq N$ intersects $\varphi(N')$ transversely.

Hamiltonian diffeomorphisms and Hofer distance. Let M be a manifold, and $H \in C^\infty\big([0,1]\times M,\mathbb{R}\big)$. We abbreviate $H_t := H(t,\cdot)$. The Hofer semi-norm of H is defined to be

$$\|H\| := \|H\|_M := \int_0^1 \big(\sup_M H_t - \inf_M H_t\big) \in [0,\infty]. \tag{14}$$

It follows from Lemma 28 in the appendix that the function $[0,1] \ni t \mapsto \sup_M H_t - \inf_M H_t \in [0,\infty]$ is Borel measurable. Therefore, the integral (14) is well-defined. Let now (M,ω) be a symplectic manifold. For a function $H \in C^\infty(M,\mathbb{R})$ we define the vector field X_H generated by H via the formula $dH = \omega(X_H,\cdot)$. For $t \in \mathbb{R}$ we denote by φ_H^t the time-t-flow of the family $(X_{H_s})_{s\in\mathbb{R}}$ (if it exists). A diffeomorphism $\varphi : M \to M$ is called Hamiltonian iff there exists a function $H \in C^\infty\big([0,1]\times M,\mathbb{R}\big)$ such that φ_H^1 exists and equals φ. We denote by $\mathrm{Ham}(M,\omega)$ the group of all Hamiltonian diffeomorphisms. For $\varphi,\psi \in \mathrm{Ham}(M,\omega)$ we define the Hofer distance $d^{M,\omega}(\varphi,\psi) = d(\varphi,\psi)$ to be

$$d(\varphi,\psi) := \inf\big\{\|H\| \,\big|\, H \in C^\infty\big([0,1]\times M,\mathbb{R}\big) :\, \varphi_H^1 = \psi^{-1}\circ\varphi\big\} \in [0,\infty]. \tag{15}$$

We denote by $\mathrm{Ham}_{\mathrm{c}}(M,\omega) \subseteq \mathrm{Ham}(M,\omega)$ the subgroup of all diffeomorphisms that are generated by some compactly supported function $H : [0,1]\times M \to \mathbb{R}$. We define the *compactly supported Hofer distance* of $\varphi,\psi \in \mathrm{Ham}_{\mathrm{c}}(M,\omega)$ to be

$$d_{\mathrm{c}}(\varphi,\psi) := \inf\big\{\|H\| \,\big|\, H \in C_{\mathrm{c}}^\infty\big([0,1]\times M,\mathbb{R}\big) :\, \psi^{-1}\circ\varphi = \varphi_H^1\big\}. \tag{16}$$

Geometric boundedness. Let (M,ω) be a symplectic manifold, and J an ω-compatible almost complex structure on M. We call (M,ω,J) (geometrically) bounded iff the Riemannian metric $g_{\omega,J} := \omega(\cdot, J\cdot)$ is complete with bounded sectional curvature and injectivity radius bounded away from 0. We call (M,ω) (geometrically) bounded iff there exists an almost complex structure J such that (M,ω,J) is bounded. Examples are closed symplectic manifolds, cotangent bundles of closed manifolds, and symplectic vector spaces. For an almost complex manifold (M,J) and a totally real submanifold $N \subseteq M$ we define

$$A_{S^2}(M,J) := \inf\left\{\int_{S^2} u^*\omega \,\middle|\, u : S^2 \to M \ J\text{-holomorphic}\right\}, \tag{17}$$

$$A_{\mathbb{D}}(M,N,J) := \inf\left\{\int_{\mathbb{D}} u^*\omega \,\middle|\, u : \mathbb{D} \to M \ J\text{-holomorphic}, \ u(S^1) \subseteq N\right\}.$$

Let (M,ω) be a bounded symplectic manifold, and $L \subseteq M$ a Lagrangian submanifold. We define

$$A_{\mathrm{b}}(M,\omega,L) := \sup\left\{\min\left\{A_{S^2}(M,J), A_{\mathbb{D}}(M,L,J)\right\}\right\}, \tag{18}$$

where the supremum is taken over all ω-compatible almost complex structures J on M such that (M,ω,J) is bounded.

3. Reduction to the Lagrangian case

The proof of Theorem 1 is based on the following result, which is a reformulation of the Main Theorem in [**Ch**]. Recall the definition (12) of $A(M,\omega,N)$.

7. Theorem. *Let (M,ω) be a bounded symplectic manifold and $L \subseteq M$ a closed Lagrangian submanifold. Then there exists a constant $C \in (0,\infty]$ such that $C \geq A(M,\omega,L)$ and the following holds. If $\varphi \in \mathrm{Ham}(M,\omega)$ satisfies*

$$d(\varphi, \mathrm{id}) < C, \tag{19}$$

and $\varphi(L) \pitchfork L$ (i.e. $\varphi(L)$ and L intersect transversely), then

$$\#(L \cap \varphi(L)) \geq \sum_i b_i(L, \mathbb{Z}_2). \tag{20}$$

For the convenience of the reader, let us recall Chekanov's theorem:

8. Theorem ([**Ch**])**.** *Let (M,ω) be a bounded symplectic manifold, $L \subseteq M$ a closed Lagrangian submanifold, and $\varphi \in \mathrm{Ham}_{\mathrm{c}}(M,\omega)$. If $d_{\mathrm{c}}(\varphi,\mathrm{id}) < A_{\mathrm{b}}(M,\omega,L)$ (defined as in (18)) and $\varphi(L) \pitchfork L$ then (20) holds.*

Proof of Theorem 7. Let M,ω and L be as in the hypothesis. We may assume without loss of generality that M and L are connected. We define $C := A_{\mathrm{b}}(M,\omega,L)$. By quantization of energy for pseudo-holomorphic spheres and disks, this number is positive.

1. Claim. *We have $C \geq A(M,\omega,L)$.*

Proof of Claim 1. Let J be as in the definition of boundedness of (M,ω). Then $A_{\mathbb{D}}(M,L,J) \geq A(M,\omega,L)$. Furthermore, let $u : S^2 \to M$ be a J-holomorphic map. Since M is connected, there exists a smooth map $v : S^2 \cong \mathbb{C} \cup \{\infty\} \to M$ that is smoothly homotopic to u and satisfies $v(\infty) \in L$. We choose a smooth map $f : \mathbb{D} \to S^2$ that maps the interior $B_1 \subseteq \mathbb{D}$ diffeomorphically and in an orientation preserving way onto $\mathbb{C}$. Then the map $v \circ f : \mathbb{D} \to M$ is smooth and satisfies $v \circ f(S^1) \subseteq L$. Furthermore,

$$\int_{S^2} u^*\omega = \int_{S^2} v^*\omega = \int_{\mathbb{D}} (v \circ f)^*\omega.$$

It follows that the set of numbers occurring in (17) is contained in $S(M,\omega,N)$ (as defined in (11)), and hence $A_{S^2}(M,J) \geq A(M,\omega,L)$. Claim 1 follows. □

Let $\varphi \in \mathrm{Ham}(M,\omega)$ be such that condition (19) is satisfied and $\varphi(L) \pitchfork L$. Applying Lemma 20 below with $K := L$ there exists a function $H \in C^\infty_{\mathrm{c}}\big([0,1] \times M, \mathbb{R}\big)$ such that (52) holds. It follows that $d_{\mathrm{c}}(\varphi^1_H, \mathrm{id}) < C$. Furthermore, by the first condition in (52) we have $\varphi^1_H(L) = \varphi(L)$, and hence $\varphi^1_H(L) \pitchfork L$. Therefore, the hypotheses of Theorem 8 are satisfied with φ replaced by φ^1_H. Inequality (20) follows from the conclusion of this theorem. This proves Theorem 7. □

Let now (M,ω) be a symplectic manifold and $N \subseteq M$ a coisotropic submanifold. Recall that N_ω denotes the set of isotropic leaves of N, and $\pi_N : N \to N_\omega$ the canonical projection. We abbreviate $\pi := \pi_N$. We define $\widetilde{M} := M \times N_\omega$, and ι_N and $\widetilde{N}$ as in (4). For a map $\varphi : M \to M$ we define

$$\widetilde{\varphi} := \varphi \times \mathrm{id}_{N_\omega} : \widetilde{M} \to \widetilde{M}. \tag{21}$$

Assume that N is regular. Then we denote by $\big(N_\omega, \mathcal{A}_{N,\omega}, \omega_N\big)$ the symplectic quotient of $(N, \omega|_N)$. We equip $\widetilde{M}$ with the manifold structure determined by the manifold structure on M and $\mathcal{A}_{N,\omega}$. Furthermore, we define $\widetilde{\omega} := \omega \oplus (-\omega_N)$. This is a symplectic form on $\widetilde{M}$.

9. Lemma. *Let (M,ω) be a symplectic manifold, and $N \subseteq M$ a connected coisotropic submanifold.*

(a) For every map $\varphi : M \to M$ we have

$$\widetilde{\varphi} \circ \iota_N(\mathrm{Fix}(\varphi, N)) = \widetilde{N} \cap \widetilde{\varphi}(\widetilde{N}).$$

Assume now also that N is regular. Then:

(b) The map ι_N is a Lagrangian embedding with respect to $\widetilde{\omega}$.

(c) If $\varphi : M \to M$ is a diffeomorphism then the pair (N,φ) is non-degenerate if and only $\widetilde{\varphi}(\widetilde{N}) \pitchfork \widetilde{N}$.

For the proof of Lemma 9 we need the following.

10. Lemma. *Let (M,ω) be a symplectic manifold, $N \subseteq M$ a regular coisotropic submanifold, $\varphi : M \to M$ a diffeomorphism, $F \subseteq N$ a leaf, and $x \in C^\infty([0,1],F)$ a path. Assume that $x(1) = \varphi \circ x(0)$, and let $v \in T_{x(0)}N \cap T_{x(0)}\varphi^{-1}(N)$. Then (13) is equivalent to $\pi_*\varphi_*v \neq \pi_*v$.*

Proof of Lemma 10. We fix $y \in N$. By Lemma 24(b) below we have $\ker d\pi(y) = T_yN^\omega$. Hence we may define

$$\Phi_y : (T_yN)_\omega = T_yN/T_yN^\omega \to T_{N_y}(N_\omega), \quad \Phi_y(w + T_yN^\omega) := d\pi(y)w.$$

Since $d\pi(y) : T_yN \to (T_yN)_\omega$ is surjective, Φ_y is an isomorphism. Furthermore, $\Phi_y\mathrm{pr}_y = d\pi(y)$, where $\mathrm{pr}_y : T_yN \to (T_yN)_\omega$ denotes the canonical projection. Hence Lemma 24(f) below implies that $d\pi(x(0)) = \Phi_{x(1)}\mathrm{hol}_x^{\omega,N}\mathrm{pr}_{x(0)}$. It follows that

$$(\pi_*\varphi_* - \pi_*)v = \Phi_{x(1)}\big(\mathrm{pr}_{x(1)}\varphi_* - \mathrm{hol}_x^{\omega,N}\mathrm{pr}_{x(0)}\big)v.$$

Since $\Phi_{x(1)}$ is an isomorphism, the statement of Lemma 10 follows. □

Proof of Lemma 9. **Statement (a)** follows from the definition of a leafwise fixed point.

We prove (b). That ι_N is an injective Lagrangian immersion follows immediately from the definitions. To see that it is an open map onto its image $\widetilde{N}$, let $U \subseteq N$ be open. We choose an open subset $V \subseteq M$ such that $V \cap N = U$, and denote by $\pi_1 : \widetilde{M} = M \times N_\omega \to M$ the projection onto the first factor. Then $\iota_N(U)$ is the intersection of $\widetilde{N}$ with the open subset $\pi_1^{-1}(V) \subseteq \widetilde{M}$, hence it is relatively open in $\widetilde{N}$. This proves (b).

We prove (c). Assume that $x \in \mathrm{Fix}(\varphi,N)$ and denote $\widetilde{x} := \widetilde{\varphi} \circ \iota_N(x)$. Note that by assertion (a) $\widetilde{x} \in \widetilde{N} \cap \widetilde{\varphi}(\widetilde{N})$.

1. Claim. *$\widetilde{N}$ and $\widetilde{\varphi}(\widetilde{N})$ intersect transversely at $\widetilde{x}$ if and only if*

$$0 \neq v \in T_xN \cap T_x\varphi^{-1}(N) \implies \pi_*\varphi_*v \neq \pi_*v.$$

Proof of Claim 1. For $y \in N$ we have

$$T_{\iota_N(y)}\widetilde{N} = \iota_{N*}T_yN = \big\{(v,\pi_*v)\,\big|\,v \in T_yN\big\}. \tag{22}$$

Setting $y := x$, it follows that

$$T_{\widetilde{x}}\widetilde{\varphi}(\widetilde{N}) = \widetilde{\varphi}_*\iota_{N*}T_xN = \big\{(\varphi_*v,\pi_*v)\,\big|\,v \in T_xN\big\}. \tag{23}$$

On the other hand, since $x \in \mathrm{Fix}(\varphi,N)$, we have $\iota_N \circ \varphi(x) = \widetilde{\varphi} \circ \iota_N(x) = \widetilde{x}$. Therefore, applying (22) with $y := \varphi(x)$, and combining with (23), we obtain

$$T_{\widetilde{x}}\widetilde{N} \cap T_{\widetilde{x}}\widetilde{\varphi}(\widetilde{N}) = \big\{\big(\varphi_*v,\pi_*v\big)\,\big|\,v \in T_xN :\, \varphi_*v \in T_{\varphi(x)}N,\, \pi_*\varphi_*v = \pi_*v\big\}.$$

Claim 1 follows from this. □

Assume now that (N,φ) is non-degenerate. Let $\widetilde{x}_0 \in \widetilde{N} \cap \widetilde{\varphi}(\widetilde{N})$. By assertion (a) there exists $x_0 \in \mathrm{Fix}(\varphi, N)$ such that $\widetilde{\varphi} \circ \iota_N(x_0) = \widetilde{x}_0$. We choose a path $x \in C^\infty([0,1], N_{x_0})$ such that $x(0) = x_0$ and $x(1) = \varphi(x_0)$. If $0 \neq v \in T_{x_0}N \cap T_{x_0}\varphi^{-1}(N)$ then by (13) and Lemma 10 we have $\pi_*\varphi_* v \neq \pi_* v$. Therefore by Claim 1 with x replaced by x_0 the manifolds $\widetilde{\varphi}(\widetilde{N})$ and $\widetilde{N}$ intersect transversely at $\widetilde{x}_0$. It follows that $\widetilde{\varphi}(\widetilde{N}) \pitchfork \widetilde{N}$.

Conversely, assume now that $\widetilde{\varphi}(\widetilde{N}) \pitchfork \widetilde{N}$. Let $F \subseteq N$ be a leaf, and $x \in C^\infty([0,1], F)$ a path, and assume that $x(1) = \varphi \circ x(0)$, and $0 \neq v \in T_{x(0)}N \cap T_{x(0)}\varphi^{-1}(N)$. By Claim 1 we obtain $\pi_*\varphi_* v \neq \pi_* v$. Therefore, by Lemma 10 the inequality (13) is satisfied. It follows that (N,φ) is non-degenerate. This proves (c) and completes the proof of Lemma 9. □

11. Lemma (Key Lemma)**.** *Let (M,ω) be a symplectic manifold, $N \subseteq M$ a closed, regular coisotropic submanifold, and let $\widetilde{M}, \widetilde{\omega}$ and $\widetilde{N}$ be defined as in (3,4). Then*

$$A(M,\omega,N) = A(\widetilde{M},\widetilde{\omega},\widetilde{N}). \tag{24}$$

Proof of Lemma 11. In order to show that (24) with "=" replaced by "≥" holds, let $u \in C^\infty(\mathbb{D}, M)$ be a map such that there exists a leaf $F \subseteq N$ satisfying $u(S^1) \subseteq F$. Then the map

$$\widetilde{u} : \mathbb{D} \to \widetilde{M} = M \times N_\omega, \quad \widetilde{u}(z) := (u(z), N_{u(1)})$$

satisfies $\int_{\mathbb{D}} \widetilde{u}^*\widetilde{\omega} = \int_{\mathbb{D}} u^*\omega$. The inequality "≥" in (24) follows.

To show the opposite inequality, let $\widetilde{u} = (v, w') \in C^\infty(\mathbb{D}, \widetilde{M})$ be a map such that $\widetilde{u}(S^1) \subseteq \widetilde{N}$. It suffices to prove that there exists a map $u \in C^\infty(\mathbb{D}, M)$ such that $u(S^1)$ is contained in an isotropic leaf of N, and

$$\int_{\mathbb{D}} u^*\omega = \int_{\mathbb{D}} \widetilde{u}^*\widetilde{\omega}. \tag{25}$$

To see this, we choose a smooth map $\rho : [0,1] \to [0,1]$ such that

$$\rho(1/2) = 1, \quad \rho(r) = r,\ \forall r \in [0,1/4], \quad \rho(r) = 1 - r,\ \forall r \in [3/4,1],$$
$$\rho'(r) > 0,\ \forall r \in (0,1/2), \quad \rho'(r) < 0,\ \forall r \in (1/2,1),$$

and all derivatives of ρ vanish at 1/2. We define $\varphi : \mathbb{D} \to \mathbb{D}$ by $\varphi(rz) := \rho(r)z$, for $r \in [0,1]$ and $z \in S^1$.

1. Claim. *There exists a smooth map $u : \mathbb{D} \to M$ such that*

$$u(z) = v \circ \varphi(z), \quad \textit{if } |z| \leq 1/2,$$

$$u(z) \in N, \quad \pi \circ u(z) = w' \circ \varphi(z), \quad \textit{if } 1/2 < |z| \leq 1. \tag{26}$$

Proof of Claim 1. We define $f' : [0,1] \times S^1 \to N_\omega$ by $f'(t,z) := w'(tz)$. For every $z \in S^1$ we have by assumption $\widetilde{u}(z) \in \widetilde{N}$, i.e. $\pi \circ v(z) = w'(z) = f'(1,z)$. Hence f' is a smooth homotopy in N_ω, ending at the map $\pi \circ v|_{S^1}$.

Since N is closed and the projection $\pi : N \to N_\omega$ is a submersion, results by Ehresmann imply that there exists a smooth map $f : [0,1] \times S^1 \to N$ such that $\pi \circ f = f'$ and $f(1,\cdot) = v|_{S^1}$. (See [**Eh**], the proposition on p. 31 and the second proposition on p. 35.) We define $u : \mathbb{D} \to M$ by

$$u(z) := \begin{cases} v \circ \varphi(z), & \text{if } |z| \le 1/2, \\ f\big(\rho(|z|), z/|z|\big), & \text{if } 1/2 < |z| \le 1. \end{cases}$$

This map has the required properties. This proves Claim 1. □

We choose a map u as in Claim 1. By the definition of $\widetilde{\omega}$ we have $\widetilde{u}^*\widetilde{\omega} = v^*\omega - {w'}^*\omega_N$. Therefore, equality (25) is a consequence of the following:

2. Claim. *We have*

$$\int_{B_{1/2}} u^*\omega = \int_{\mathbb{D}} v^*\omega, \quad \int_{\mathbb{D}\setminus B_{1/2}} u^*\omega = -\int_{\mathbb{D}} {w'}^*\omega_N. \tag{27}$$

Proof of Claim 2. The first identity in (27) follows from the fact that φ restricts to a diffeomorphism from $B_{1/2}$ onto B_1. To prove the second identity, observe that by the definition of the symplectic form ω_N on the quotient N_ω, and (26), we have on $\mathbb{D} \setminus B_{1/2}$,

$$u^*\omega = u^*\pi^*\omega_N = (\pi \circ u)^*\omega_N = (w' \circ \varphi)^*\omega_N = \varphi^*{w'}^*\omega_N. \tag{28}$$

Since φ restricts to an orientation reversing diffeomorphism from $B_1 \setminus \bar{B}_{1/2}$ onto $B_1 \setminus \{0\}$, (28) implies that $\int_{B_1\setminus\bar{B}_{1/2}} u^*\omega = -\int_{B_1\setminus\{0\}} {w'}^*\omega_N$. This implies the second identity in (27). This proves Claim 2 and completes the proof of Lemma 11. □

4. Proofs of the main results

Proof of Theorem 1. Let M, ω and N be as in the hypothesis of this theorem. Without loss of generality we may assume that N is connected. Since N is regular, the symplectic quotient $\big(N_\omega, \mathcal{A}_{N,\omega|_N}, \omega_N\big)$ of $(N, \omega|_N)$ is well-defined. We define $\widetilde{M}, \widetilde{\omega}$ and $\widetilde{N}$ as in (3,4). Since N is closed, N_ω is closed. By a straight-forward argument the product of two bounded symplectic manifolds is bounded. It follows that $(\widetilde{M}, \widetilde{\omega}) = \big(M \times N_\omega, \omega \oplus (-\omega_N)\big)$ is bounded. Furthermore, by Lemma 9(b) $\widetilde{N} \subseteq \widetilde{M}$ is a Lagrangian submanifold. It is closed since N is closed. Therefore, applying Theorem 7 with M, ω replaced by $\widetilde{M}, \widetilde{\omega}$, and $L := \widetilde{N}$, there exists a positive constant $C \ge A(\widetilde{M}, \widetilde{\omega}, \widetilde{N})$ such that the statement of that theorem holds. We check that this constant has the required properties: By Lemma 11, we have $C \ge A(M, \omega, N)$. Let now $\varphi \in \mathrm{Ham}(M, \omega)$ be such that (N, φ) is non-degenerate and inequality (1) is satisfied. We define $\widetilde{\varphi}$ as in (21). Using Lemma 19 below, it follows that

$$d^{\widetilde{M},\widetilde{\omega}}(\widetilde{\varphi}, \mathrm{id}_{\widetilde{M}}) \le d^{M,\omega}(\varphi, \mathrm{id}) < C.$$

Furthermore, by non-degeneracy of (N,φ) and Lemma 9(c) we have $\widetilde{\varphi}(\widetilde{N}) \pitchfork \widetilde{N}$. Therefore, by the conclusion of Theorem 7, we have

$$\big|\big(\widetilde{N}\cap\widetilde{\varphi}(\widetilde{N})\big)\big| \geq \sum_i b_i(\widetilde{N},\mathbb{Z}_2). \tag{29}$$

On the other hand, parts (a) and (b) of Lemma 9 imply that

$$\big|\mathrm{Fix}(\varphi,N)\big| = \big|\big(\widetilde{N}\cap\widetilde{\varphi}(\widetilde{N})\big)\big|, \quad b_i(\widetilde{N},\mathbb{Z}_2) = b_i(N,\mathbb{Z}_2),\ \forall i.$$

Combining this with (29), inequality (2) follows. This proves Theorem 1. □

Proof of Theorem 2. Let M,ω,N,f,ι and ε be as in the hypothesis. We choose a Riemannian metric g on M, and denote by $|\cdot|$, ℓ and d the norm on T_xM (for $x\in M$), the length functional, and the distance function, all with respect to g. Furthermore, for $x,y\in M$, a linear map $T:T_xM\to T_yM$, and a bilinear map $b:T_xM\times T_xM\to\mathbb{R}$ we define

$$\begin{aligned} |T|_{\mathrm{op}} &:= \max\big\{|Tv|\,\big|\,v\in T_xM,\ |v|=1\big\},\\ |b| &:= \max\big\{|b(v,w)|\,\big|\,v,w\in T_xM,\ |v|=|w|=1\big\}.\end{aligned}$$

Assume first that there exists a closed presymplectic manifold (M',ω') and an ω'-horizontal distribution $H\subseteq TM'$ such that N is the zero-section of $(TM'^{\omega'})^*$, M is an open neighborhood of N, and $\omega=\Omega_{\omega',H}$ (as defined in (10) with ω replaced by ω'). We identify M' with N. We denote by $\pi:(TN^\omega)^*\to N$ the canonical projection. We choose a smooth function $\rho:M\to\mathbb{R}$ that has compact support and equals 1 in a neighborhood of N. We define $F:=\rho\cdot(f\circ\pi):M\to\mathbb{R}$. Let $t\in\mathbb{R}$, and $x_0\in\mathrm{Crit}f\subseteq N$. Then $dF(x_0)=df(x_0)d\pi(x_0)=0$, hence $X_F(x_0)=0$, and therefore $x_0\in\mathrm{Fix}(\varphi_F^t,N)$. It follows that $\mathrm{Crit}f\subseteq\mathrm{Fix}(\varphi_F^t,N)$.

1. Claim. *There exists a number $t_1>0$ such that for every $t\in(0,t_1]$, we have* $\mathrm{Fix}(\varphi_F^t,N)\subseteq\mathrm{Crit}f$.

Proof of Claim 1. For $(t_0,x_0)\in[0,\infty)\times M$ we abbreviate

$$\ell(t_0,x_0) := \ell\big([0,t_0]\ni t\mapsto\varphi_X^t(x_0)\in M\big).$$

2. Claim. *There exists a constant C such that for every $t\in[0,\infty)$, and $x_0\in\mathrm{Fix}(\varphi_F^t,N)$, we have*

$$d(x_0,\varphi_F^t(x_0)) \leq C\ell(t,x_0)^2. \tag{30}$$

Proof of Claim 2. We denote by $g|_N$ the restriction of g to $TN\oplus TN$. Applying Proposition 23 below with $\mathcal{F}$ the isotropic foliation of N, the horizontal distribution H, and M,g replaced by $N,g|_N$, there exists a constant C such that the conclusion of that lemma holds. Let $t\in[0,\infty)$ and $x_0\in\mathrm{Fix}(\varphi_F^t,N)$. We define $x:[0,t]\to N$ by $x(s):=\pi\circ\varphi_F^s(x_0)$. By Lemma 21 below with M,ω replaced by $N,\omega|_N$, we have $\dot{x}(s)=\pi_*X_F\circ\varphi_F^s(x_0)\in H_{x(s)}$, for every $s\in[0,t]$. Furthermore, $x(t)=\varphi_F^t(x_0)$ lies in the isotropic leaf

through $x(0)$, since $x(0) = x_0 \in \mathrm{Fix}(\varphi_F^t, N)$. Therefore, the conditions of Proposition 23 are satisfied, and hence

$$d(x_0, \varphi_F^t(x_0)) \leq d^{g|_N}(x_0, \varphi_F^t(x_0)) \leq C\ell(x)^2. \tag{31}$$

Here $d^{g|_N}$ denotes the distance function on N induced by $g|_N$. We denote by $K \subseteq M$ the support of ρ, and $C' := \max_{y\in K} |d\pi(y)|_{\mathrm{op}}$. Then $\ell(x) \leq C'\ell(t, x_0)$. Combining this with (31), we obtain

$$d(x_0, \varphi_F^t(x_0)) \leq CC'^2\ell(t, x_0)^2.$$

Claim 2 follows. □

We choose a constant C as in Claim 2. We apply Lemma 26 below with $X := X_F$, and f replaced by the map $[0,\infty) \ni a \mapsto Ca \in [0,\infty)$, and we choose a constant $t_1 := \varepsilon > 0$ as in the assertion of that lemma. Let $t \leq t_1$ and $x_0 \in \mathrm{Fix}(\varphi_F^t, N)$. Then inequality (30) holds, and therefore by the conclusion of Lemma 26, $X_F(x_0) = 0$. It follows that $df(x_0)d\pi(x_0) = dF(x_0) = \omega(X_F(x_0), \cdot) = 0$, and therefore $df(x_0) = 0$, i.e. $x_0 \in \mathrm{Crit} f$. This proves Claim 1. □

We choose a number t_1 as in Claim 1. Since F has compact support, there exists a number $t_2 > 0$ so small that $\left\|\iota\circ\varphi_F^t\circ\iota^{-1} - \mathrm{id}\right\|_{C^1(\iota(M))} < \varepsilon$, for every $t \in [0, t_2]$.

3. Claim. *There exists $t_3 > 0$ such that for every $0 < t \leq t_3$ the pair (N, φ_F^t) is non-degenerate.*

Proof of Claim 3. Let $x_0 \in \mathrm{Crit} f$. Then $X_F(x_0) = 0$, and hence the derivative $dX_F(x_0) : T_{x_0}M \to T_{x_0}M$ is well-defined. We fix $t \in \mathbb{R}$. Then $\varphi_F^t(x_0) = x_0$, and hence $d\varphi_F^t(x_0)$ is a linear map from $T_{x_0}M$ to $T_{x_0}M$. Hence we may define

$$T_{x_0}^t := d\varphi_F^t(x_0) - \mathrm{id} - t dX_F(x_0) : T_{x_0}M \to T_{x_0}M. \tag{32}$$

We have $T_{x_0}^0 = 0$. Furthermore, by a calculation in local coordinates, we have $\frac{d}{dt}\big|_{t=0} d\varphi_F^t(x_0) = dX_F(x_0)$. Hence by Taylor's theorem there exists a constant C_{x_0} such that $|T_{x_0}^t|_{\mathrm{op}} \leq C_{x_0}t^2$, for every $t \in [0, t_1]$.

A calculation in Darboux charts shows that the bilinear form $B_{x_0} : T_{x_0}M \times T_{x_0}M \ni (v, w) \mapsto \omega\big(dX_F(x_0)v, w\big) \in \mathbb{R}$ is the Hessian of F. Since $F|_N = f$, it follows that the restriction $b_{x_0} := B_{x_0}|_{T_{x_0}N\times T_{x_0}N}$ is the Hessian of f. We define the linear map $A : T_{x_0}N \to T_{x_0}N$ by $g_{x_0}(\cdot, A\cdot) := b_{x_0}$, and we denote by V_+ and V_- the direct sum of the positive and negative eigenspaces of A, respectively. It follows that A is self-adjoint with respect to g_{x_0}. Since by assumption f is Morse, the form b_{x_0} is non-degenerate, hence A is an isomorphism, and therefore $T_{x_0}N = V_+ \oplus V_-$. We define

$c_{x_0} := \min\big\{|\lambda|\,|\,\lambda \text{ eigenvalue of } A\big\}$. Since f is Morse the set $\mathrm{Crit} f$ is isolated. Since N is compact, it follows that $\mathrm{Crit} f$ is finite. Hence we may define

$$t_3 := \min\left(\left\{\frac{c_{x_0}}{2C_{x_0}|\omega_{x_0}|}\,\Big|\, x_0 \in \mathrm{Crit} f\right\} \cup \{t_1\}\right). \tag{33}$$

For $x_0 \in N$ we denote by $\mathrm{pr}_{x_0} : T_{x_0}N \to T_{x_0}N/T_{x_0}N^\omega$ the canonical projection. Let $0 < t \le t_3$, $F \subseteq N$ be a leaf, $x \in C^\infty([0,1], F)$ a path, and $v \in T_{x(0)}N \cap T_{x(0)}(\varphi_F^t)^{-1}(N)$. Assume that $\varphi_F^t(x(0)) = x(1)$, and that

$$\mathrm{hol}_x^{\omega,N}\mathrm{pr}_{x(0)}v = \mathrm{pr}_{x(1)}d\varphi_F^t(x(0))v. \tag{34}$$

Claim 3 is a consequence of the following.

4. Claim. *We have $v = 0$.*

Proof of Claim 4. Since $\varphi_F^t(x_0) = x(1) \in F$, we have $x_0 \in \mathrm{Fix}(\varphi_F^t, N)$. Therefore, using $t \le t_3 \le t_1$, Claim 1 implies that $x_0 \in \mathrm{Crit} f$, and hence $x(1) = x(0)$. Recall that π_N denotes canonical projection from N to the set of isotropic leaves N_ω. We abbreviate $x_0 := x(0)$. Since by assumption N is regular, it follows from (34) and Lemma 10 that $d\pi_N(x_0)d\varphi_F^t(x_0)v = d\pi_N(x_0)v$. By Lemma 24(b) this means that

$$d\varphi_F^t(x_0)v - v \in \ker d\pi(x_0) = T_{x_0}N^\omega. \tag{35}$$

We define $v_\pm \in V_\pm$ by $v_+ + v_- := v$. Since A is g_{x_0}-self-adjoint, eigenvectors of A for distinct eigenvalues are g_{x_0}-orthogonal to each other. It follows that $b_{x_0}(v_-, v_+) = 0$, and therefore,

$$c_{x_0}|v_\pm|^2 \le \big|b_{x_0}(v, v_\pm)\big| = \big|\omega\big(dX_F(x_0)v, v_\pm\big)\big|. \tag{36}$$

By (32) and (35) we have

$$t\omega\big(dX_F(x_0)v, v_\pm\big) + \omega(T_{x_0}^t v, v_\pm) = \omega\big(d\varphi_F^t(x_0)v - v, v_\pm\big) = 0. \tag{37}$$

Furthermore, we may estimate

$$\big|\omega(T_{x_0}^t v, v_\pm)\big| \le |\omega_{x_0}||T_{x_0}^t|_{\mathrm{op}}|v||v_\pm|. \tag{38}$$

Consider the case $|v_+| \ge |v_-|$. Then $|v| \le \sqrt{2}|v_+|$, since V_+ and V_- are orthogonal with respect to g_{x_0}. Hence (36,37,38) imply that $c_{x_0}|v_+|^2 t \le \sqrt{2}|\omega_{x_0}||T_{x_0}^t|_{\mathrm{op}}|v_+|^2$. Combining this with the inequalities $|T_{x_0}^t|_{\mathrm{op}} \le C_{x_0}t^2$ and $t \le t_3$, and (33), we obtain $c_{x_0}|v_+|^2 t \le (1/\sqrt{2})c_{x_0}|v_+|^2 t$. Since $c_{x_0}, t > 0$, it follows that $0 = |v_+| \ge |v_-|$, and therefore $v = v_+ + v_- = 0$. The case $|v_-| \ge |v_+|$ is treated in an analogous way. This proves Claim 4 and completes the proof of Claim 3. □

We choose a number t_3 as in Claim 3, and define $\varphi := \varphi_F^{\min\{t_1,t_2,t_3\}}$. This map has the required properties. This proves Theorem 2 in the case in which M and N are an open neighborhood and the zero-section of $(TM'^{\omega'})^*$, for some closed presymplectic manifold (M', ω'), and ω is of the form $\Omega_{\omega',H}$.

Consider now the general case. Let M, ω, N, f, ι and ε be as in the hypothesis of Theorem 2. We denote by $\widetilde{N} \subseteq (TN^\omega)^*$ the zero section. We choose an ω-horizontal distribution $H \subseteq TN$, and define $\widetilde{\omega} := \Omega_{\omega|_N, H}$. By a theorem by C.-M. Marle there exist compact neighborhoods $K \subseteq M$ of N and $\widetilde{K} \subseteq (TN^\omega)^*$ of $\widetilde{N}$, and a diffeomorphism $\psi : K \to \widetilde{K}$ such that $\psi(N) = \widetilde{N}$ and $\psi^*\widetilde{\omega} = \omega$. (See 4.5. Théorème on p. 79 in [**Ma**].) We define

$$\widetilde{f} := f \circ \psi|_N^{-1} : \widetilde{N} \to \mathbb{R}, \quad \widetilde{\iota} := \iota \circ \psi^{-1} : \widetilde{K} \to \mathbb{R}^{2n}.$$

We denote by $\mathrm{int}\widetilde{K}$ the interior of $\widetilde{K}$. We proved that there exists $\widetilde{\varphi} \in \mathrm{Ham_c}(\mathrm{int}\widetilde{K}, \widetilde{\omega})$ such that $\mathrm{Fix}(\widetilde{\varphi}, \widetilde{N}) = \mathrm{Crit}\widetilde{f}$, $\big(\widetilde{N}, \widetilde{\varphi}, \widetilde{\omega}\big)$ is non-degenerate, and $\big\|\widetilde{\iota} \circ \widetilde{\varphi} \circ \widetilde{\iota}^{-1} - \mathrm{id}\big\|_{C^1(\widetilde{\iota}(\mathrm{int}\widetilde{K}))} < \varepsilon$. We define $\varphi : M \to M$ to be the extension of $\psi^{-1} \circ \widetilde{\varphi} \circ \psi : \mathrm{int}K \to \mathrm{int}K$ by the identity. It follows that $\varphi \in \mathrm{Ham_c}(M, \omega)$,

$$\mathrm{Fix}(\varphi, N) = \psi^{-1}(\mathrm{Fix}(\widetilde{\varphi}, \widetilde{N})) = \mathrm{Crit}(\widetilde{f} \circ \psi|_N) = \mathrm{Crit}f,$$

and (N, φ, ω) is non-degenerate. Furthermore, $\iota \circ \varphi \circ \iota^{-1} = \widetilde{\iota} \circ \widetilde{\varphi} \circ \widetilde{\iota}^{-1}$ on $\iota(K) = \widetilde{\iota}(\widetilde{K}) \subseteq \mathbb{R}^{4n}$, and therefore,

$$\big\|\iota \circ \varphi \circ \iota^{-1} - \mathrm{id}\big\|_{C^1(\iota(M))} = \big\|\widetilde{\iota} \circ \widetilde{\varphi} \circ \widetilde{\iota}^{-1} - \mathrm{id}\big\|_{C^1(\widetilde{\iota}(\widetilde{K}))} < \varepsilon.$$

Hence φ satisfies the required properties. This proves Theorem 2 in the general case. $\square$

Proof of Proposition 3. Note that the isotropic leaf through a point $\Theta \in V(k,n)$ is the orbit $\mathrm{U}(k)\cdot\Theta$ of the action of $U(k)$ on $V(k,n)$ by multiplication from the left.

1. Claim. *We have* $A\big(\mathbb{C}^{k\times n}, \omega_0, V(k,n)\big) \leq \pi$.

Proof of Claim 1. Consider the map $u : \mathbb{D} \to \mathbb{C}^{k\times n}$ defined by ${u^1}_1(z) := z$, for $z \in \mathbb{D}$, ${u^i}_i \equiv 1$, for $i = 2, \ldots, k$, and ${u^i}_j \equiv 0$, for $i = 1, \ldots, k$, $j = 1, \ldots, n$ such that $i \neq j$. Then $u(S^1)$ is contained in the leaf $\mathrm{U}(k)\cdot u(1)$. Furthermore, $\int_{\mathbb{D}} u^*\omega_0 = \pi$. Hence $\pi \in S\big(\mathbb{C}^{k\times n}, \omega_0, V(k,n)\big)$ (defined as in (11)). Claim 1 follows from this. $\square$

2. Claim. *We have* $A\big(\mathbb{C}^{k\times n}, \omega_0, V(k,n)\big) \geq \pi$.

Proof of Claim 2. Let $u \in C^\infty(\mathbb{D}, \mathbb{C}^{k\times n})$, and assume that $\int_{\mathbb{D}} u^*\omega_0 > 0$, and that there exists an isotropic leaf $F \subseteq V(k,n)$ such that $u(z) \in F$, for every $z \in S^1$. Since the action of $U(k)$ on $V(k,n)$ is free, there exists a unique map $g_0 : S^1 \to U(k)$ such that $u(z) = g_0(z)u(1)$, for $z \in S^1$. This map is smooth. We define d to be the degree of the map $\det \circ g_0 : S^1 \to S^1$. Claim 2 is a consequence of the following.

3. Claim. *We have* $\int_{\mathbb{D}} u^*\omega_0 = d\pi$.

Proof of Claim 3. We define

$$h_0 : S^1 \to U(k), \quad h_0(z) := \mathrm{diag}(z^d, 1, \dots, 1) g_0(z)^{-1}.$$

Here $\mathrm{diag}(a_1, \dots, a_k)$ means the diagonal $k \times k$-matrix with diagonal entries $a_1, \dots, a_k$. The map $\det \circ h_0 : S^1 \to S^1$ has degree 0. Since the determinant induces an isomorphism of the fundamental groups of $U(k)$ and S^1, it follows that there exists a continuous homotopy from the constant map $\mathbf{1}$ to h_0. This gives rise to a continuous map $h : \mathbb{D} \to U(k)$ such that $h|_{S^1} = h_0$. We may assume w.l.o.g. that h is smooth. Let $\mu : \mathbb{C}^{k \times n} \to \mathrm{Lie}\,\mathrm{U}(k)$ be a moment map for the action of $U(k)$ on $V(k,n)$. By a straight-forward calculation, we have

$$(hu)^* \omega_0 = u^* \omega_0 - d\langle \mu \circ u, h^{-1} dh \rangle,$$

see Lemma 9 in [**Zi2**]. By Stokes' theorem, it follows that

$$\int u^* \omega_0 = \int_{\mathbb{D}} (hu)^* \omega_0. \tag{39}$$

We define the map $v : \mathbb{D} \to \mathbb{C}^{k \times n}$ by $v(z) := \mathrm{diag}(z^d, 1, \dots, 1) u(1)$, for $z \in \mathbb{D}$. Then for $z \in S^1$, we have $(hu)(z) = (h_0 u)(z) = v(z)$. Therefore,

$$\int_{\mathbb{D}} (hu)^* \omega_0 = \int_{\mathbb{D}} v^* \omega_0 = d\pi \sum_{j=1,\dots,n} \left| {u^1}_j(1) \right|^2 = d\pi.$$

Here in the last equality we used that the first row of $u(1)$ has norm 1. This proves Claim 3 and hence Claim 2. □

Claims 1 and 2 imply that $A\big(\mathbb{C}^{k \times n}, \omega_0, V(k,n)\big) = \pi$. This proves the first assertion. To prove the second assertion, let $C > \pi$. Then there exists $\widetilde{H} \in C^\infty_{\mathrm{c}}(\mathbb{C}, \mathbb{R})$ such that $\varphi^1_{\widetilde{H}}(\mathbb{D}) \cap \mathbb{D} = \emptyset$, and $\|\widetilde{H}\| < C$. (See for example the proof of Proposition 1.4. in [**Ho**].) We define $\pi : \mathbb{C}^{k \times n} \to \mathbb{C}$ by $\pi(\Theta) := {\Theta^1}_1$, and choose $\rho \in C^\infty(\mathbb{C}^{k \times n}, [0,1])$ with compact support, such that $\rho = 1$ on $\bigcup_{t \in [0,1]} \varphi^t_{\widetilde{H} \circ \pi}(V(k,n))$. We define $H := \rho \cdot (\widetilde{H} \circ \pi) : \mathbb{C}^{k \times n} \to \mathbb{R}$ and $\varphi := \varphi^1_H$. Then $\varphi \in \mathrm{Ham}_{\mathrm{c}}(\mathbb{C}^{k \times n}, \omega_0)$, and $d_{\mathrm{c}}(\varphi, \mathrm{id}) \le \|H\| = \|\widetilde{H}\| < C$. Furthermore, $\pi(V(k,n)) = \mathbb{D}$ and $\pi \circ \varphi^1_{\widetilde{H} \circ \pi} = \varphi^1_{\widetilde{H}} \circ \pi$. It follows that

$$\pi(V(k,n)) \cap \pi \circ \varphi^1_{\widetilde{H} \circ \pi}(V(k,n)) \subseteq \mathbb{D} \cap \varphi^1_{\widetilde{H}}(\mathbb{D}) = \emptyset.$$

Hence $V(k,n) \cap \varphi^1_{\widetilde{H} \circ \pi}(V(k,n)) = \emptyset$. Our choice of ρ implies that $\varphi^1_{\widetilde{H} \circ \pi} = \varphi^1_H$ on $V(k,n)$. It follows that $V(k,n) \cap \varphi^1_H(V(k,n)) = \emptyset$. Therefore, φ has the required properties. This proves Proposition 3. □

Proof of Proposition 4. Let $M_i, \omega_i, L, M, \omega, N$ and ψ be as in the hypothesis of the Conjecture. Without loss of generality we may assume that M_1 and

L are connected. We define

$$\widetilde{M} := M_1 \times M_2 \times M_1, \quad \widetilde{\omega} := \omega_1 \oplus \omega_2 \oplus (-\omega_1),$$
$$\widetilde{N} := \big\{(x_1, x_2, x_1) \in M \,\big|\, x_1 \in M_1,\, x_2 \in L\big\}, \quad \widetilde{\varphi} := \varphi \times \mathrm{id}_{M_1}.$$

Then $\widetilde{N}$ is a Lagrangian submanifold of $\widetilde{M}$. Since M_1 and L are closed manifolds by assumption, it follows that $\widetilde{N}$ is closed. The map

$$\widetilde{\psi} : \widetilde{M} \to \widetilde{M}, \quad \widetilde{\psi}(x_1, x_2, y) := (y, \psi(x_2), x_1).$$

is an $\widetilde{\omega}$-anti-symplectic involution whose fixed point set equals $\widetilde{N}$. Furthermore, $N = M_1 \times L \subseteq M = M_1 \times M_2$ is a regular ω-coisotropic submanifold, and the symplectic quotient $\big(N_\omega, \mathcal{A}_{N,\omega|_N}, \omega_N\big)$ of $(N, \omega|_N)$ is isomorphic to (M_1, ω_1) via the map $M_1 \ni y \mapsto \{y\} \times L \in N_\omega$. Via this map, the definitions of $\widetilde{M}, \widetilde{\omega}$ and $\widetilde{\varphi}$ agree with (3,4,21). Hence by non-degeneracy of (N, φ) and Lemma 9(c), we have $\widetilde{N} \pitchfork \widetilde{\varphi}(\widetilde{N})$. Thus $\widetilde{M}, \widetilde{\omega}$ and $\widetilde{N}$ satisfy the hypotheses of the Lagrangian AGC. Supposing that this conjecture is true, it follows that

$$\big|\widetilde{N} \cap \widetilde{\varphi}(\widetilde{N})\big| \geq \sum_i b_i(\widetilde{N}, \mathbb{Z}_2). \tag{40}$$

The manifolds N and $\widetilde{N}$ are diffeomorphic, and therefore their Betti sums agree. Furthermore, Lemma 9(a) implies that $\big|\mathrm{Fix}(\varphi, N)\big| = \big|\widetilde{N} \cap \widetilde{\varphi}(\widetilde{N})\big|$. Combining this with (40), we obtain inequality (2). This proves Proposition 4. □

Proof of Corollary 5. Let (M, ω) be a bounded and aspherical symplectic manifold, and (M', ω') a closed and regular presymplectic manifold of corank $\dim M - \dim M'$. Assume that M' has a simply-connected isotropic leaf F_0, and that there exists an embedding $\psi : M' \to M$ satisfying $\psi^*\omega = \omega'$. Let $\varphi \in \mathrm{Ham}(M, \omega)$. It suffices to prove that $N := \psi(M')$ intersects $\varphi(N)$. Replacing M' by the connected component of M' containing F_0, we may assume without loss of generality that M' is connected. It follows from Proposition 12 below that the submanifold $\psi(M') \subseteq M$ is coisotropic. We choose a constant $C > 0$ as in Theorem 1. If (N, φ) is degenerate, then by definition $\mathrm{Fix}(\varphi, N) \neq \emptyset$, and hence $N \cap \varphi(N) \neq \emptyset$. So assume that (N, φ) is non-degenerate.

1. Claim. *We have $A(M, \omega, N) = \infty$.*

Proof of Claim 1. Let $u \in C^\infty(\mathbb{D}, M)$ be a smooth map such that there exists $F \in N_\omega$ satisfying $u(S^1) \subseteq F$. Since N is closed and the canonical projection $\pi_N : N \to N_\omega$ is a submersion, by the proposition on p. 31 in [**Eh**] it is a locally trivial fiber bundle. Since N is connected, it follows that F is diffeomorphic to F_0, and therefore simply connected. Hence there exists a smooth map $v : \mathbb{D} \to F$ such that u and v agree on the boundary

S^1. We choose a map $\rho \in C^\infty([0,1],[0,1])$ such that $\rho(r) = r$ for $r \leq 1/2$, $\rho(1) = 1$, $\rho'(r) > 0$, for every $r \in (0,1)$, and all derivatives of ρ vanish at $r = 1$. We define $f : \mathbb{D} \to \mathbb{D}$ by $f(rz) := \rho(r)z$, for $r \in [0,1]$ and $z \in S^1$. We denote by $\overline{\mathbb{D}}$ the disk with the reversed orientation, and by $w := (u \circ f)\#(v \circ f) : S^2 \cong \mathbb{D}\#\overline{\mathbb{D}} \to M$ the concatenation of the maps $u \circ f$ and $v \circ f$. This map is smooth, and since (M,ω) is symplectically aspherical, we have

$$0 = \int_{S^2} w^*\omega = \int_{B_1} (u \circ f)^*\omega - \int_{\mathbb{D}} (v \circ f)^*\omega. \tag{41}$$

Since v takes values in the isotropic leaf F, we have $(v \circ f)^*\omega = 0$. Thus (41) implies that $\int_{\mathbb{D}} u^*\omega = \int_{B_1} (u \circ f)^*\omega = 0$. Claim 1 follows from this. $\square$

Claim 1 implies that $C = \infty$, and hence inequality (1) holds. Therefore, the conditions of Theorem 1 are satisfied. Applying this theorem, inequality (2) follows, and therefore $N \cap \varphi(N) \neq \emptyset$. This proves Corollary 5. $\square$

Appendix A. Auxiliary results

A.1. (Pre-)symplectic geometry. The following result was used in Section 1 and in the proof of Corollary 5. It will also be needed for the proof of Proposition 15 below.

12. Proposition. *Let (M,ω) be a presymplectic manifold, M' a manifold, $\psi : M' \to M$ an immersion, and $\omega' := \psi^*\omega$. If ω' has constant corank then the inequality*

$$\dim M' + \operatorname{corank} \omega' \leq \dim M + \operatorname{corank} \omega \tag{42}$$

holds. Suppose now that ψ is an embedding. Then $\psi(M') \subseteq M$ is coisotropic if and only if ω' has constant corank and equality in (42) holds.

For the proof of this proposition we need the following lemma.

13. Lemma. *Let (V,ω) and (V',ω') be presymplectic vector spaces (possibly ∞-dimensional), and $\Psi : V' \to V$ a linear map such that $\Psi^*\omega = \omega'$. Then*

$$\dim V' + \operatorname{corank} \omega' \leq \dim V + \operatorname{corank} \omega + \dim \ker \Psi + \dim \ker \Psi|_{V'^{\omega'}}. \tag{43}$$

Furthermore, if $\dim V, \dim V' < \infty$ then $\Psi V' \subseteq V$ is coisotropic if and only if equality in (43) holds.

The proof of this lemma is based on the following.

14. Lemma. *Let (V,ω) be a presymplectic vector space, and $W \subseteq V$ a subspace. Then*

$$\dim W + \dim W^\omega \leq \dim V + \dim V^\omega. \tag{44}$$

Furthermore, if $\dim V < \infty$ and $V^\omega \subseteq W$ then equality in (44) holds.

Proof of Lemma 14. To see that (44) holds, we define the linear map $\omega^\# : V \to V^*$ by $\omega^\# v := \omega(v,\cdot)$. We denote by $i_W : W \to V$ the inclusion. Then $W^\omega = \ker(i_W^*\omega^\#)$, and therefore,

$$\dim \operatorname{im}(i_W^*\omega^\#) + \dim W^\omega = \dim V. \tag{45}$$

Consider the canonical isomorphism $\iota : V \to V^{**}$, $\iota(v)(\varphi) := \varphi(v)$. A direct calculation shows that $(\omega^\#)^*\iota = -\omega^\#$. It follows that $(\omega^\# i_W)^*\iota = -i_W^*\omega^\#$, and therefore

$$\dim \operatorname{im}(\omega^\# i_W) = \dim \operatorname{im}(\omega^\# i_W)^* = \dim \operatorname{im}(i_W^*\omega^\#). \tag{46}$$

On the other hand, we have $\dim\ker(\omega^\# i_W) \leq \dim\ker(\omega^\#) = \dim V^\omega$. Combining this with (46), we obtain

$$\dim W = \dim\ker(\omega^\# i_W) + \dim \operatorname{im}(\omega^\# i_W) \leq \dim V^\omega + \dim \operatorname{im}(i_W^*\omega^\#).$$

This together with (45) implies (44).

Assume now that $V^\omega \subseteq W$. Then $\dim\ker(\omega^\# i_W) = \dim\ker(\omega^\#)$, and therefore the above argument shows that equality in (44) holds. This proves Lemma 14. □

Proof of Lemma 13. The hypothesis $\Psi^*\omega = \omega'$ implies that

$$\Psi(V'^{\omega'}) \subseteq (\Psi V')^\omega. \tag{47}$$

It follows that

$$\dim V'^{\omega'} = \dim \Psi(V'^{\omega'}) + \dim\ker \Psi|_{V'^{\omega'}} \leq \dim(\Psi V')^\omega + \dim\ker\Psi|_{V'^{\omega'}}, \tag{48}$$

and therefore

$$\begin{aligned} \dim V' + \operatorname{corank}\omega' &= \dim(\Psi V') + \dim\ker\Psi + \dim V'^{\omega'} \\ &\leq \dim(\Psi V') + \dim(\Psi V')^\omega \\ &\quad + \dim\ker\Psi + \dim\ker\Psi|_{V'^{\omega'}}. \end{aligned} \tag{49}$$

Applying Lemma 14, inequality (43) follows. The second statement is a consequence of the following two claims.

1. Claim. *$\Psi V'$ is coisotropic if and only if equality in (47) holds.*

Proof of Claim 1. If equality in (47) holds then $(\Psi V')^\omega = \Psi(V'^{\omega'}) \subseteq \Psi V'$, hence $\Psi V'$ is coisotropic. Conversely, if $\Psi V'$ is coisotropic then a straightforward argument implies that $(\Psi V')^\omega \subseteq \Psi(V'^{\omega'})$, hence equality in (47) holds. This proves Claim 1. □

Assume now that $\dim V, \dim V' < \infty$.

2. Claim. *Equality in (47) holds if and only if equality in (43) holds.*

Proof of Claim 2. Suppose that equality in (47) holds. Then equality in (48) and in (49) holds. Furthermore, $V^\omega \subseteq (\Psi V')^\omega = \Psi(V'^{\omega'}) \subseteq \Psi V'$, and hence by Lemma 14

$$\dim(\Psi V') + \dim(\Psi V')^\omega = \dim V + \operatorname{corank}\omega.$$

Combining this with (49), it follows that equality in (43) holds.

Assume now on the contrary that equality in (43) holds. Then

$$\begin{aligned}\dim(\Psi V')^\omega &\leq \dim V + \dim V^\omega - \dim \Psi V' \\ &= \dim V' - \dim \Psi V' - \dim\ker\Psi + \dim V'^{\omega'} - \dim\ker\Psi|_{V'^{\omega'}} \\ &= 0 + \dim\Psi(V'^{\omega'}).\end{aligned}$$

Here in the first step we used Lemma 14. It follows that equality in (47) holds. This proves Claim 2 and concludes the proof of Lemma 13. □

Proof of Proposition 12. Let M, ω, M', ψ and ω' be as in the hypothesis. We choose a point $x' \in M'$, and define

$$V' := T_{x'}M',\ \Omega' := \omega'|_{V'\times V'},\ V := T_{\psi(x')}M,\ \Omega := \omega|_{V\times V},\ \Psi := d\psi(x').$$

Then the hypotheses of Lemma 13 are satisfied with ω, ω' replaced by Ω, Ω'. It follows that inequality (43) holds. Since Ψ is injective, this implies inequality (42), provided that ω' has constant corank.

Suppose now that ψ is an embedding. Assume that ω' has constant corank and equality in (42) holds. Let $x' \in M'$. Applying Lemma 13 with V, V' and Ψ as above, and ω, ω' replaced by Ω, Ω', it follows that $T_{\psi(x')}\psi(M') = \Psi V' \subseteq T_{\psi(x')}M$ is coisotropic. Hence $\psi(M') \subseteq M$ is coisotropic. Conversely, assuming that $\psi(M') \subseteq M$ is coisotropic, Lemma 13 implies that the corank of ω' at any point $x' \in M'$ equals $\dim M + \operatorname{corank}\omega - \dim M'$. This proves Proposition 12. □

The next result was used in Section 1. Let (X, σ) be a closed symplectic manifold, $\pi : E \to X$ a closed smooth fiber bundle, $H \subseteq TE$ a horizontal subbundle, and let $N, \pi_X, \iota_H, \omega := \Omega_{\sigma,H}$ and M be as in the construction explained in that section, on p. 3.

15. Proposition. *$N \subseteq M$ is a regular coisotropic submanifold. Furthermore, if (X, σ) is symplectically aspherical then $A(M, \omega, N) = \infty$.*

For the proof of Proposition 15 we need the following. We denote by i_E the embedding of E as the zero section $N \subseteq V^*E$.

16. Lemma. *We have $\pi^*\sigma = i_E^*\omega$.*

Proof of Lemma 16. We denote by j_E the embedding of E as the zero-section of T^*E. Then $\iota_H \circ i_E = j_E$, and therefore, denoting by λ_{can} the canonical one-form on T^*E, we obtain $i_E^*\iota_H^*\omega_{\mathrm{can}} = -dj_E^*\lambda_{\mathrm{can}} = -d0 = 0$.

Since $\pi_X \circ i_E = \pi$, it follows that $\pi^*\sigma = i_E^*\big(\pi_X^*\sigma + \iota_H^*\omega_{\mathrm{can}}\big) = i_E^*\omega$. This proves Lemma 16. □

For any manifold M and any positive integer k we denote by $\Omega^k(M)$ the space of differential forms of degree k.

17. Lemma. *Let M and N be smooth manifolds, $k \geq 1$, $\omega \in \Omega^k(N)$ a closed form, and $u : [0,1] \times M \to N$ a smooth map such that $u(t,x) = u(0,x)$, for every $t \in [0,1]$, $x \in \partial M$. Then there exists $\alpha \in \Omega^{k-1}(M)$ such that $d\alpha = u(1,\cdot)^*\omega - u(0,\cdot)^*\omega$, and $\alpha(x) = 0$, for all $x \in \partial M$.*

Proof of Lemma 17. This follows from the proof of Theorem VI(7.13) p. 270 in the book [**Bo**]. □

18. Remark. *Let (M,ω) and (M',ω') be presymplectic manifolds, and $\psi : M' \to M$ a diffeomorphism such that $\psi^*\omega = \omega'$. Then by a straightforward argument the image of every isotropic leaf of M' under ψ is an isotropic leaf of M. Furthermore, the map $M'_{\omega'} \ni F' \mapsto \psi(F') \in M_\omega$ is a bijection.*

Proof of Proposition 15. We may assume without loss of generality that E is connected. By Lemma 16 we have $i_E^*\omega = \pi^*\sigma$. Furthermore, for every $e \in E$ we have $T_eE^{\pi^*\sigma} = \ker d\pi(e)$, and hence $i_E^*\omega$ has constant corank equal to the dimension of the fiber of E. It follows that equality in (42) holds with $M' := E$, $\psi := i_E$, and $\omega' := \psi^*\omega$. Hence by Proposition 12 the submanifold $N \subseteq M$ is coisotropic. Furthermore, the leaf relation of $(E,\pi^*\sigma)$ consists of all pairs $(x_0',x_1') \in E \times E$ that lie in the same connected component of one of the fibers of E. It follows from an argument involving local trivializations for E that this is a closed subset and a submanifold. Hence $(E,\pi^*\sigma)$ is regular. Since $\omega|_N$ is the push-forward of $\pi^*\sigma$ under the diffeomorphism $i_E : E \to N$, it follows that N is regular.

To prove the second statement, assume that (X,σ) is aspherical. Let $u \in C^\infty(\mathbb{D},M)$ be a map such that there exists a leaf $F \in N_\omega$ satisfying $u(S^1) \subseteq F$. It suffices to prove that $\int_{\mathbb{D}} u^*\omega = 0$. To see this, we denote by π_0 the canonical projection from V^*E to its zero-section N. We choose a smooth function $\rho : [0,1] \to [0,1]$ such that $\rho(r) = r$, for $r \leq 1/3$, and $\rho(r) = 1$, for $r \geq 2/3$. We define $u_0 : \mathbb{D} \to M$ by $u_0(rz) := \pi_0 \circ u(\rho(r)z)$, for $r \in [0,1]$ and $z \in S^1$.

1. Claim. *We have*

$$\int_{\mathbb{D}} u^*\omega = \int_{\mathbb{D}} u_0^*\omega. \tag{50}$$

Proof of Claim 1. We define the map $h : [0,1] \times V^*E \to V^*E$ by $h(t,e,\alpha) := (e,t\alpha)$, and the map $f : [0,1] \times \mathbb{D} \to V^*E$ by

$$f(t,rz) := h\Big(t, u\big(\big(tr + (1-t)\rho(r)\big)z\big)\Big),\ \forall r \in [0,1],\ z \in S^1.$$

Observe that $f(0,\cdot) = u_0$, and $f(1,\cdot) = u$. Since $u(S^1) \subseteq F \subseteq N$, we have $f(t,z) = f(0,z)$, for every $t \in [0,1]$ and $z \in S^1$. Hence the hypotheses of Lemma 17 are satisfied, with M, N replaced by $\mathbb{D}, V^*E$, and u replaced by the map f. If follows that there exists $\alpha \in \Omega^1(\mathbb{D})$ such that

$$d\alpha = u^*\omega - u_0^*\omega, \quad \alpha(z) = 0, \forall z \in S^1.$$

Together with Stokes' Theorem this implies (50). This proves Claim 1. □

2. Claim. *We have* $\int_{\mathbb{D}} u_0^*\omega = 0$.

Proof of Claim 2. To see this, we choose an orientation preserving diffeomorphism $\varphi : \mathbb{C} \to B_1$, and we define the map $f : S^2 \cong \mathbb{C} \cup \{\infty\} \to X$ by

$$f(z) := \begin{cases} \pi_X \circ u_0 \circ \varphi(z), & \text{if } z \in \mathbb{C}, \\ \pi_X \circ u(1), & \text{if } z = \infty. \end{cases}$$

3. Claim. *This map is smooth.*

Proof of Claim 3. $f|_{\mathbb{C}}$ is smooth. Furthermore, by Remark 18 there exists a $\pi^*\sigma$-isotropic leaf F' of E such that $i_E(F') = F$. Let $e_0 \in F'$. Since for every $e \in E$ we have $T_eE^{\pi^*\sigma} = \ker d\pi(e)$, it follows that F' is the connected component of the fiber of E containing e_0. Since $\pi_X \circ i_E = \pi$, this implies that π_X equals the constant $\pi(e_0) \in X$ on F. We choose a number $r_0 > 0$ such that $|\varphi(rz)| \geq 2/3$, for $r \geq r_0$ and $z \in S^1$. Let $z \in \mathbb{C} \setminus B_{r_0}$. Then $u_0 \circ \varphi(z) \in u(S^1) \subseteq F$, and therefore $f(z) = \pi(e_0)$. Since also $f(\infty) = \pi(e_0)$, it follows that f is smooth on S^2. This proves Claim 3. □

By Claim 3 and symplectic asphericity of X we have

$$0 = \int_{S^2} f^*\sigma = \int_{\mathbb{C}} \big(\pi_X \circ u_0 \circ \varphi\big)^*\sigma = \int_{B_1} (\pi_X \circ u_0)^*\sigma. \tag{51}$$

Let $v_0 : \mathbb{D} \to E$ be the unique map such that $i_E \circ v_0 = u_0$. Then $\pi_X \circ u_0 = \pi \circ v_0$, and hence using Lemma 16,

$$(\pi_X \circ u_0)^*\sigma = v_0^*\pi^*\sigma = v_0^* i_E^*\omega = u_0^*\omega.$$

Inserting this into (51), Claim 2 follows. □

Claims 1 and 2 imply that $\int_{\mathbb{D}} u^*\omega = 0$. It follows that $A(M,\omega,N) = \infty$. This proves the second statement and completes the proof of Proposition 15. □

The next lemma was used in the proof of Theorem 1.

19. Lemma. *Let* (M,ω) *and* (M',ω') *be symplectic manifolds,* $\varphi, \psi \in \mathrm{Ham}(M,\omega)$*, and* $\varphi', \psi' \in \mathrm{Ham}(M',\omega')$*. Then*

$$d^{M\times M',\omega\oplus\omega'}\big(\varphi\times\varphi', \psi\times\psi'\big) \leq d^{M,\omega}(\varphi,\psi) + d^{M',\omega'}(\varphi',\psi').$$

Proof of Lemma 19. If $H : [0,1] \times M \to \mathbb{R}$ and $H' : [0,1] \times M' \to \mathbb{R}$ are smooth Hamiltonians generating $\psi^{-1} \circ \varphi$ and ${\psi'}^{-1} \circ \varphi'$ respectively, then the function

$$\widetilde{H} : [0,1] \times \widetilde{M} \to \mathbb{R}, \quad \widetilde{H}(t,x,x') := H(t,x) + H'(t,x')$$

generates $(\psi^{-1} \circ \varphi) \times ({\psi'}^{-1} \circ \varphi')$. Furthermore, we have $\sup_{\widetilde{M}} \widetilde{H}(t,\cdot,\cdot) = \sup_M H(t,\cdot) + \sup_{M'} H'(t,\cdot)$, for every $t \in [0,1]$, and similarly for the infimum. It follows that $\|\widetilde{H}\|_{M \times M', \omega \oplus \omega'} = \|H\|_{M,\omega} + \|H'\|_{M',\omega'}$. The statement of Lemma 19 follows from this. $\square$

The next lemma was used in the proof of Theorem 7.

20. Lemma. *Let (M,ω) be a symplectic manifold, $\varphi \in \mathrm{Ham}(M,\omega)$, $K \subseteq M$ a compact subset, and $C > d(\varphi, \mathrm{id})$ a constant. Then there exists a function $H \in C^\infty_{\mathrm{c}}\big([0,1] \times M, \mathbb{R}\big)$ such that*

$$\varphi^1_H = \varphi \textit{ on } K, \quad \|H\| < C. \tag{52}$$

Proof of Lemma 20. Let M, ω, φ, K and C be as in the hypothesis. We choose a smooth function $\widetilde{H} : [0,1] \times M \to \mathbb{R}$ that generates φ and satisfies $\|\widetilde{H}\| < C$. We also fix an open neighborhood $U \subseteq M$ of K with compact closure, and we define $K' := \bigcup_{t \in [0,1]} \varphi^t_{\widetilde{H}}(\bar{U})$. We choose an open neighborhood $V \subseteq M$ of K' such that $\bar{V}$ is compact. It follows from a C^∞-version of Urysohn's Lemma for $\mathbb{R}^n$ (see for example Theorem 1.1.3 p.4 in [**KP**]) and a partition of unit argument that there exists a smooth function $f : M \to [0,1]$ such that $f^{-1}(0) = M \setminus V$ and $f^{-1}(1) = K'$. We fix a point $x_0 \in M$ and define

$$H : [0,1] \times M \to \mathbb{R}, \quad H(t,x) := f(x)\big(\widetilde{H}(t,x) - \widetilde{H}(t,x_0)\big).$$

Then the support of H is contained in $\bar{V}$ and hence compact. Furthermore, for $t \in [0,1]$ and $x \in \bar{U}$ we have $H(t, \varphi^t_{\widetilde{H}}(x)) = \widetilde{H}(t, \varphi^t_{\widetilde{H}}(x)) - \widetilde{H}(t,x_0)$. It follows that $X_{H(t,\cdot)}(x) = X_{\widetilde{H}(t,\cdot)}(x)$, for $x \in K'$. This implies that $\varphi^1_H(x) = \varphi^1_{\widetilde{H}}(x) = \varphi(x)$, for $x \in \bar{U}$, and therefore the first condition in (52) holds. Finally, observe that

$$\max_{x \in M} H(t,x) \le \sup_{x \in M} \widetilde{H}(t,x) - \widetilde{H}(t,x_0).$$

Combining this with a similar inequality for $\min_{x \in M} H(t,x)$, it follows that $\|H\| \le \|\widetilde{H}\|$. Since $\|\widetilde{H}\| < C$, the second condition in (52) follows. This proves Lemma 20. $\square$

The next lemma was used in the proof of Theorem 2. Let (M,ω) be a closed presymplectic manifold, and $H \subseteq TM$ an ω-horizontal distribution. We denote by $\pi : (TM^\omega)^* \to M$ the canonical projection. We define ι_H and $\Omega_{\omega,H}$ as in (9,10).

21. Lemma. *Let $f \in C^\infty(M,\mathbb{R})$, $U \subseteq (TM^\omega)^*$ be an open neighborhood of the zero section on which $\Omega_{\omega,H}$ is non-degenerate, and X the Hamiltonian vector field on U generated by $f \circ \pi : M \to \mathbb{R}$ with respect to $\Omega_{\omega,H}$. Then $\pi_* X(x) \in H_{\pi(x)}$, for every $x \in U$.*

Proof of Lemma 21. We fix $x := (y,\alpha) \in U$, and denote by $\mathrm{pr}^H : T_yM \to T_yM^\omega$ the canonical projection along H. We denote by $\pi' : T^*M \to M$ the canonical projection, and by $i : (T_yM^\omega)^* \to T_x(TM^\omega)^*$ and $i' : T_y^*M \to T_{\iota_H(x)}T^*M$ the canonical inclusions.

1. Claim. *For every $v \in T_xU$ and $\beta \in (T_yM^\omega)^*$, we have*

$$\Omega_{\omega,H}(v, i\beta) = \beta \mathrm{pr}^H \pi_* v.$$

Proof of Claim 1. We have $\pi_* i\beta = 0$, and

$$\iota_{H*} i\beta = \frac{d}{dt}\bigg|_{t=0} \iota_H\big(y, \alpha + t\beta\big) = \frac{d}{dt}\bigg|_{t=0} \big(y, (\alpha+t\beta)\mathrm{pr}^H\big) = i'(\beta\mathrm{pr}^H).$$

It follows that

$$\begin{aligned} \Omega_{\omega,H}(v, i\beta) &= \omega\big(\pi_* v, \pi_* i\beta\big) + \omega_{\mathrm{can}}\big(\iota_{H*}v, \iota_{H*} i\beta\big) \\ &= 0 + \omega_{\mathrm{can}}\big(\iota_{H*}v, i'(\beta\mathrm{pr}^H)\big) \\ &= \beta\mathrm{pr}^H \pi'_* \iota_{H*} v \\ &= \beta\mathrm{pr}^H \pi_* v. \end{aligned}$$

This proves Claim 1. □

Claim 1 implies that

$$0 = f_* \pi_* i\beta = d(f\circ\pi)(x) i\beta = \Omega_{\omega,H}(X(x), i\beta) = \beta\mathrm{pr}^H \pi_* X(x),$$

for every $\beta \in (T_yM^\omega)^*$. It follows that $\mathrm{pr}^H \pi_* X(x) = 0$, i.e. $\pi_* X(x) \in H_y$. This proves Lemma 21. □

A.2. Foliations. In this subsection Proposition 6 is proved. This result was used to define the linear holonomy of a foliation. The second result of this subsection is an estimate for the distance between the initial and end point of a path x in a foliation, provided that these points lie in the same leaf, and x is tangent to a given horizontal distribution. For the proof of Proposition 6 we need the following lemma. Let $(M,\mathcal{F})$ be a foliated manifold and $(U,\varphi) \in \mathcal{F}$ a chart. We write $\varphi =: (\varphi^\xi, \varphi^\eta) : U \to \mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$.

22. Lemma. *Let $F \subseteq M$ be a leaf of $\mathcal{F}$, $a \le b$, and $u : [a,b] \to F \cap U$ a continuous map. Then $\varphi^\xi \circ u : [a,b] \to \mathbb{R}^{n-k}$ is locally constant.*

Proof of Lemma 22. Let $(M,\mathcal{F})$ be a foliated manifold. By definition, the leaf topology on F is the topology $\tau_F^{\mathcal{F}}$ generated by the sets $\varphi^{-1}(\{0\}\times\mathbb{R}^k)$, where $(U,\varphi) \in \mathcal{F}$ is such that $\varphi^{-1}(\{0\}\times\mathbb{R}^k) \subseteq F$. It is second countable,

see for example Lemma 1.3. on p. 11 in the book [**Mol**]. It follows that there exists a countable collection of surjective foliation charts $\varphi_i : U_i \to \mathbb{R}^n$ ($i \in \mathbb{N}$), such that $\big(\varphi_i^{-1}(\{0\} \times \mathbb{R}^k)\big)_{i\in\mathbb{N}}$ is a basis for $\tau_F^{\mathcal{F}}$. Let $(U, \varphi) \in \mathcal{F}$. Then $U \cap F \in \tau_F^{\mathcal{F}}$, and therefore there exists a subset $S \subseteq \mathbb{N}$ such that $U \cap F = \bigcup_{i\in S} U_i$. For each $i \in S$ compatibility of φ and φ_i implies that φ^ξ is constant on U_i. It follows that $\varphi^\xi(U \cap F) \subseteq \mathbb{R}^{n-k}$ is at most countable. The statement of Lemma 22 follows from this. □

Proof of Proposition 6. Let $M, \mathcal{F}, F, a, b, x, N$ and y_0 be as in the hypothesis. To prove **statement (a)**, let $T : T_{y_0}N \to T_{x(a)}M$ be a linear map.

1. Claim. *There exists a smooth map $f : N \to M$ such that $f(y_0) = x(a)$ and $df(y_0) = T$.*

Proof of Claim 1. We choose a quadruple (U, V, φ, ψ), where $U \subseteq M$ and $V \subseteq N$ are neighborhoods of $x(a)$ and y_0, respectively, and $\varphi : T_{x(a)}M \to U$ and $\psi : V \to T_{y_0}N$ are diffeomorphisms, such that the following holds. Identifying $T_0(T_{x(a)}M) = T_{x(a)}M$ and $T_0(T_{y_0}N) = T_{y_0}N$, we have

$$\varphi(0) = x(a), \quad d\varphi(0) = \mathrm{id}_{T_{x(a)}M}, \quad \psi(y_0) = 0, \quad d\psi(y_0) = \mathrm{id}_{T_{y_0}N}.$$

Furthermore, we choose a function $\rho \in C^\infty(T_{y_0}N, [0, 1])$ such that $\rho = 1$ in a neighborhood of 0, and $\rho = 0$ outside some compact subset of $T_{y_0}N$. We define $f(y) := \varphi \circ (\rho \cdot T) \circ \psi(y)$ for $y \in V$, and $f(y) := y_0$, for $y \in N \setminus V$. This map has the required properties. This proves Claim 1. □

We denote by $\pi_2 : [a, b] \times M \to M$ the projection onto the second factor.

2. Claim. *There exists a smooth section $s : [a, b] \times M \to \pi_2^* T\mathcal{F}$ of compact support, such that $s(t, x(t)) = \dot{x}(t)$, for every $t \in [a, b]$.*

Proof of Claim 2. For every $t \in [a, b]$ we choose a foliation chart $\varphi_t : U_t \to \mathbb{R}^n$, such that $U_t \subseteq M$ is an open neighborhood of $x(t)$. Shrinking U_t and reparametrizing φ_t, we may assume that φ_t is surjective. We choose a finite subset $S \subseteq [a, b]$ such that $x([a, b]) \subseteq U := \bigcup_{t\in S} U_t$. We fix $t \in S$, and define

$$A_t := \big\{\big(t', \varphi_t \circ x(t')\big) \,\big|\, x(t') \in U_t\big\}.$$

Since φ_t is surjective, A_t is a closed subset of $[a, b] \times \mathbb{R}^n$. We choose a smooth extension $f_t : [a, b] \times \mathbb{R}^n \to \mathbb{R}^k$ of the map $A_t \ni \big(t', \varphi_t \circ x(t')\big) \mapsto (\varphi_t^\eta)_* \dot{x}(t') \in \mathbb{R}^k$. We also fix a partition of unity $(\rho_t)_{t\in S}$ for U, subordinate to $(U_t)_{t\in S}$, and a smooth map $\rho : U \to [0, 1]$ with compact support, such that $\rho|_{x([a,b])} = 1$. We define

$$s : [a, b] \times M \to \pi_2^* T\mathcal{F},\ s(t', x') := \rho(x') \sum_{t\in S} \rho_t(x') d\varphi_t(x')^{-1}\big(0, f_t(t', \varphi_t(x'))\big).$$

Here each summand on the right hand side is defined to be 0 if $x' \notin U_t$. The map s has the required properties. This proves Claim 2. □

We choose a map f and a section s as in Claims 1 and 2. Since s has compact support, there exists a unique solution $u:[a,b]\times N\to M$ of the equations

$$\partial_t u(t,y)=s(t,y),\quad u(a,y)=f(y),\quad \forall t\in[a,b],\, y\in N.$$

This map has the required properties. This proves (a).

To prove **statement (b)**, let u and u' be as in the hypothesis. Consider

$$S:=\big\{t\in[a,b]\,\big|\,\mathrm{pr}^{\mathcal F}d(u(t,\cdot))(y_0)=\mathrm{pr}^{\mathcal F}d(u'(t,\cdot))(y_0)\big\}.$$

By (7) this set contains a. Furthermore, it is a closed subset of $[a,b]$.

3. Claim. *S is open.*

Proof of Claim 3. Let $t_0\in S$. We choose a chart $(U,\varphi)\in\mathcal F$ such that $x(t_0)\in U$, and a number $\varepsilon>0$ such that $x\big([t_0-\varepsilon,t_0+\varepsilon]\cap[a,b]\big)\subseteq U$. We define

$$V:=\big\{y\in N\,\big|\,u(t,y),u'(t,y)\in U,\,\forall t\in[t_0-\varepsilon,t_0+\varepsilon]\cap[a,b]\big\}.$$

This is an open subset of N. Furthermore, by the first condition in (5) we have $y_0\in V$. Let $x_0\in U$. Then by definition, the map $d\varphi^\xi(x_0):T_{x_0}M\to\mathbb R^{n-k}$ is surjective and has kernel $T_{x_0}\mathcal F$. It follows there exists a unique linear isomorphism $\Phi_{x_0}:N_{x_0}\mathcal F=T_{x_0}M/T_{x_0}\mathcal F\to\mathbb R^{n-k}$ satisfying $\Phi_{x_0}\mathrm{pr}^{\mathcal F}_{x_0}=d\varphi^\xi(x_0)$. We fix $t\in[t_0-\varepsilon,t_0+\varepsilon]$. By Lemma 22, we have

$$\varphi^\xi\circ u(t,y)=\varphi^\xi\circ u(t_0,y),\quad \varphi^\xi\circ u'(t,y)=\varphi^\xi\circ u'(t_0,y),$$

for every $y\in V$. It follows that on TV, we have

$$d\varphi^\xi(x(t))d(u(t,\cdot))=d\varphi^\xi(x(t_0))d(u(t_0,\cdot))=\Phi_{x(t_0)}\mathrm{pr}^{\mathcal F}_{x(t_0)}d(u(t_0,\cdot)),$$
$$d\varphi^\xi(x(t))d(u'(t,\cdot))=\Phi_{x(t_0)}\mathrm{pr}^{\mathcal F}_{x(t_0)}d(u'(t_0,\cdot)).$$

Since $t_0\in S$, this implies that $d\varphi^\xi(x(t))d(u(t,\cdot))(y_0)=d\varphi^\xi(x(t))d(u'(t,\cdot))(y_0)$. Using the equality $\Phi^{-1}_{x(t)}d\varphi^\xi(x(t))=\mathrm{pr}^{\mathcal F}_{x(t)}$, it follows that $\mathrm{pr}^{\mathcal F}d(u(t,\cdot))(y_0)=\mathrm{pr}^{\mathcal F}d(u'(t,\cdot))(y_0)$. Hence S is open. This proves Claim 3. □

Using Claim 3, it follows that $S=[a,b]$. This proves (b) and completes the proof of Proposition 6. □

The next result was used in the proof of Theorem 2.

23. Proposition. *Let M be a closed manifold, $\mathcal F$ a regular foliation on M, $H\subseteq TM$ an $\mathcal F$-horizontal distribution, and g a Riemannian metric on M. Then there exists a constant C such that for every $t\ge0$ and every $x\in C^\infty([0,t],M)$ the following holds. If $\dot x(s)\in H_{x(s)}$, for every $s\in[0,t]$, and $x(t)\in\mathcal F_{x(0)}$, then $d(x(0),x(t))\le C\ell(x)^2$. Here ℓ and d denote the length and distance functions with respect to g respectively.*

Proof of Proposition 23. We denote by n and k the dimension of M and of the leaves of $\mathcal F$ respectively.

1. Claim. *There exists a finite atlas $\mathcal{A}$ of surjective foliation charts $\varphi : U \to \mathbb{R}^n$, such that $\bigcup_{(U,\varphi)\in\mathcal{A}} \varphi^{-1}(B_1) = M$, and for every $(U,\varphi) \in \mathcal{A}$ and $x \in U$ the set $\varphi(\mathcal{F}_x \cap U)$ is connected.*

Proof of Claim 1. Since by assumption $\mathcal{F}$ is regular it follows from Lemma 24(e) below that there exists a smooth structure on the set M' of leaves of $\mathcal{F}$, such that the canonical projection $\pi : M \to M'$ is a submersion. Since M is closed, a result by Ehresmann implies that π is a fiber bundle. (See the proposition on p. 31 in [**Eh**].) Let $x \in M$. We choose a local trivialization $\psi_x : U'_x \times \mathcal{F}_x \to M$ of π, where U'_x is an open subset of M', such that $x \in \psi_x(U'_x \times \mathcal{F}_x)$. By combining ψ_x with charts of M' and $\mathcal{F}_x$ containing the points $\pi(x)$ and x, respectively, we obtain a foliation chart (U_x, φ_x) for M such that $x \in U_x$ and $\varphi_x^i(y) = \varphi_x^i(x)$, for every $y \in U_x \cap \mathcal{F}_x$, $i = n-k+1, \ldots, n$. By shrinking the domain and target of φ_x and rescaling, we may assume w.l.o.g. that $\varphi_x(U_x) = \mathbb{R}^n$. By compactness of M there exists a finite subset $S \subseteq M$ such that $\bigcup_{x\in S} \varphi_x^{-1}(B_1) = M$. The set $\mathcal{A} := \big\{(U_x, \varphi_x) \,|\, x \in S\big\}$ has the required properties. This proves Claim 1. □

We choose an atlas $\mathcal{A}$ as in Claim 1. For $(U,\varphi) \in \mathcal{A}$ we define ε_φ to be the distance (with respect to g) between $\varphi^{-1}(\bar{B}_1)$ and $M \setminus \varphi^{-1}(B_2)$, and we set $\varepsilon := \min\{\varepsilon_\varphi \,|\, (U,\varphi) \in \mathcal{A}\}$. Let $(U,\varphi) \in \mathcal{A}$. We define the map $\alpha_\varphi : \mathbb{R}^n \to \mathbb{R}^{k\times(n-k)}$ as follows. Namely, for $x \in \mathbb{R}^n$ we define $\alpha_\varphi(x)$ to be the unique real $k \times (n-k)$ matrix satisfying $\big\{(w, \alpha_\varphi(x)w) \,|\, w \in \mathbb{R}^{n-k}\big\} = \varphi_* H_{\varphi^{-1}(x)}$. Since H is horizontal, α is well-defined. We denote by $|\cdot|_0$ the standard norm on Euclidian space, by $|v|$ the norm of vector $v \in TM$ with respect to g, and by $\pi_1 : \mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}^{n-k}$ the canonical projection onto the first component. We choose a constant C such that

$$|v|_0 \le C\big|d(\varphi^{-1})(x)v\big|, \tag{53}$$

$$d\big(\varphi^{-1}(x), \varphi^{-1}(y)\big) \le C|x-y|_0, \tag{54}$$

$$\big|(d\alpha_\varphi(x)v)w\big|_0 \le C|v|_0\,|w|_0, \tag{55}$$

for every $(U,\varphi) \in \mathcal{A}$, $x, y \in \bar{B}_2$, $v \in \mathbb{R}^n$, and $w \in \mathbb{R}^{n-k}$. Let $t_0 \in [0,\infty)$, and $x \in C^\infty([0,t_0], M)$ a path such that $\dot{x}(t) \in H_{x(t)}$, for every $t \in [0,t_0]$, and $x(t_0) \in \mathcal{F}_{x(0)}$.

2. Claim. *If $\ell(x) \le \varepsilon$ then $d(x(0), x(t_0)) \le 2C^3\ell(x)^2$.*

Proof of Claim 2. Assume that $\ell(x) \le \varepsilon$. We choose a chart $(U,\varphi) \in \mathcal{A}$ such that $x(0) \in \varphi^{-1}(B_1)$. By the choice of ε it follows that $x(t) \in \varphi^{-1}(B_2)$, for every $t \in [0,t_0]$. Hence we may define $(a,b) := \varphi \circ x : [0,t_0] \to \mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$. By the choice of $\mathcal{A}$, the set $\varphi(\mathcal{F}_{x(0)} \cap U)$ is connected, hence it equals $\{a(0)\} \times \mathbb{R}^k$. Since $x(t_0) \in \mathcal{F}_{x(0)}$, it follows that $a(t_0) = a(0)$. Therefore,

$$\big|(a,b)(t_0) - (a,b)(0)\big|_0 = |b(t_0) - b(0)|_0. \tag{56}$$

By the definition of α_φ and the hypothesis $\dot{x}(t) \in H_{x(t)}$, we have

$$b(t_0) - b(0) = \int_0^{t_0} \dot{b}(t)dt = \int_0^{t_0} \alpha_\varphi \circ (a,b)(t)\dot{a}(t)dt. \tag{57}$$

We define $(u,v) : [0,1] \times [0,t_0] \to \mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$ by

$$(u,v)(s,t) := (a,b)(0) + s\big((a,b)(t) - (a,b)(0)\big).$$

Then $(u,v)(1,t) = (a,b)(t)$ and $\partial_t u(0,t) = 0$, for every $t \in [0,t_0]$, and therefore

$$\begin{aligned}
&\int_0^{t_0} \alpha_\varphi \circ (a,b)(t)\dot{a}(t)dt \\
&= \int_0^{t_0}\int_0^1 \partial_s\big(\alpha_\varphi \circ (u,v)\partial_t u\big)\,ds\,dt \\
&= \int_0^{t_0}\int_0^1 \Big(\partial_s\big(\alpha_\varphi \circ (u,v)\big)\partial_t u - \partial_t\big(\alpha_\varphi \circ (u,v)\big)\partial_s u \\
&\qquad\qquad + \partial_t\big(\alpha_\varphi \circ (u,v)\partial_s u\big)\Big)\,ds\,dt \\
&= \int_0^{t_0}\int_0^1 \Big(\big(d\alpha_\varphi\,\partial_s(u,v)\big)\partial_t u - \big(d\alpha_\varphi\,\partial_t(u,v)\big)\partial_s u\Big)\,ds\,dt \\
&\quad + \int_0^1 \alpha_\varphi \circ (u,v)\partial_s u\,ds\Big|_{t=0}^{t_0}.
\end{aligned} \tag{58}$$

Since $a(t_0) = a(0)$, we have $\partial_s u(s,t) = 0$, for $s \in [0,1]$ and $t = 0, t_0$. Therefore, the last term in (58) vanishes. Using (57) and (55), it follows that

$$\begin{aligned}
|b(t_0) - b(0)|_0 &\le C\int_0^{t_0}\int_0^1 \big(|\partial_s(u,v)|_0|\partial_t u|_0 + |\partial_t(u,v)|_0|\partial_s u|_0\big)\,ds\,dt \\
&\le 2C\int_0^{t_0}\int_0^1 |\partial_s(u,v)|_0|\partial_t(u,v)|_0\,ds\,dt \\
&\le 2C\int_0^{t_0} \big|(a,b)(t) - (a,b)(0)\big|_0 \left|\frac{d}{dt}(a,b)(t)\right|_0 dt \\
&\le 2C^2 \max\Big\{\big|(a,b)(t) - (a,b)(0)\big|_0 \,\Big|\, t \in [0,t_0]\Big\}\ell(x) \\
&\le 2C^2 \int_0^{t_0} \left|\frac{d}{dt}(a,b)(t)\right|_0 dt\,\ell(x) \\
&\le 2C^3\ell(x)^2.
\end{aligned} \tag{59}$$

Here in the fourth and the last step we used (53). Combining (54,56,59), we obtain

$$d(x(t_0), x(0)) \le C\big|(a,b)(t_0) - (a,b)(0)\big|_0 \le 2C^3\ell(x)^2.$$

This proves Claim 2. □

Note that in the case $\ell(x) > \varepsilon$ we have $d(x(0), x(t_0)) \leq \varepsilon^{-1}\ell(x)^2$. Combining this with Claim 2, we obtain $d(x(0), x(t_0)) \leq \max\{2C^3, \varepsilon^{-1}\}\ell(x)^2$. This concludes the proof of Proposition 23. $\square$

A.3. Further auxiliary results.

24. Lemma (Smooth structures on quotients)**.** *Let M be a set with a smooth structure and R an equivalence relation on M. Then the following holds.*

(a) There is at most one smooth structure on $M' := M/R$ such that the quotient map $\pi : M \to M'$ is a submersion.

Assume now that R is the leaf relation of some foliation $\mathcal{F}$ on M. Then:

(b) If there is a smooth structure on M' as in (a) *then* $\ker d\pi(x) = T_x\mathcal{F}$*, for every $x \in M$.*

(c) R is a closed subset of $M \times M$ if and only if M' (equipped with the quotient topology) is Hausdorff.

Assume that R is the leaf relation of some foliation $\mathcal{F}$ and the induced topology on M is Hausdorff and second countable. Then:

(d) M' is second countable.

(e) The following conditions are equivalent.

(i) There exists a smooth structure on M' as in (a)*.*

(ii) R is a submanifold of $M \times M$.

(f) Assume that there is a smooth structure $\mathcal{A}$ on M' as in (a)*. Let F be a leaf of $\mathcal{F}$, $x \in C^\infty([0,1], F)$, and $v_i \in T_{x(i)}M$, for $i = 0, 1$, be such that $\mathrm{pr}^{\mathcal{F}} v_1 = \mathrm{hol}^{\mathcal{F}}_x \mathrm{pr}^{\mathcal{F}} v_0$. Then $d\pi(x(0))v_0 = d\pi(x(1))v_1$, where the differentials are defined with respect to $\mathcal{A}$.*

25. Remark. *Let M be a set with a smooth structure and R an equivalence relation on M. We denote now by $\pi_1 : R \to M$ the projection onto the first factor. Then by a theorem by Godement, condition (i) of part* (e) *above holds if and only if (ii) is satisfied and π_1 is a submersion. (See for example Theorem 3.5.25 in the book* [**AMR**]*.)*

Proof of Lemma 24. Let M be a set with a smooth structure, and R an equivalence relation on M. **Statement (a)** follows from Proposition 3.5.21(iii) in the book [**AMR**].

Assume now that R is the leaf relation of some foliation $\mathcal{F}$ on M.

In order to prove **(b)**, let $F \in M'$ be a leaf. Since F is a regular value of π, the Implicit Function Theorem implies that $\pi^{-1}(F) = F \subseteq M$ is a submanifold, and $T_xF = \ker d\pi(x)$, for every $x \in F$. On the other hand, it follows from the definitions that $T_xF = T_x\mathcal{F}$. This proves (b).

To see **(c)**, observe that the map π is open. This follows for example from the corollary on p. 19 in [**Mol**]. (The proof goes through if M is not

Hausdorff or second countable.) Therefore, (c) follows from an elementary argument, see for example Lemma 2.3 p. 60 in [**Bo**].

Assume now also that the topology on M is Hausdorff and second countable. Using openness of π, **statement (d)** follows from Lemma 2.4 p. 60 in [**Bo**]. Furthermore, by Remark 25, **(e)** is a consequence of the following.

1. Claim. *The projection π_1 is a submersion.*

Proof of Claim 1. Let $(x_0,x_1)\in R$. We choose a path $x\in C^\infty([0,1],\mathcal{F}_{x_0})$ such that $x(i)=x_i$, for $i=0,1$. We set $a:=0$, $b:=1$, $N:=M$, $y_0:=x_0$, and $T:=\mathrm{id}_{T_{x_0}M}$. Applying Proposition 6(a) there exists a map $u\in C^\infty([0,1]\times M,M)$ such that the conditions (5) and (6) hold. By (5) the map $f:M\to M\times M$ defined by $f(y):=\big(u(0,y),u(1,y)\big)$ takes values in R and satisfies $f(x_0)=(x_0,x_1)$. Equality (6) implies that

$$d\pi_1(x_0,x_1)df(x_0)=d(\pi_1\circ f)(x_0)=d(u(0,\cdot))(x_0)=\mathrm{id}_{T_{x_0}M}.$$

It follows that $d\pi_1(x_0,x_1)$ is surjective, hence π_1 is a submersion. This proves Claim 1.

We show **(f).** We choose a map $u\in C^\infty\big([a,b]\times N_{x(a)}\mathcal{F},M\big)$ as in the definition (8) of $\mathrm{hol}^{\mathcal{F}}_x$. Then for every $s\in\mathbb{R}$, we have

$$\pi\circ u(1,\mathrm{pr}^{\mathcal{F}}sv_0)=\pi\circ u(0,\mathrm{pr}^{\mathcal{F}}sv_0).$$

Differentiating this identity with respect to s, we obtain

$$\pi_*u(1,\cdot)_*\mathrm{pr}^{\mathcal{F}}v_0=\pi_*u(0,\cdot)_*\mathrm{pr}^{\mathcal{F}}v_0. \tag{60}$$

On the other hand, the equality $\mathrm{pr}^{\mathcal{F}}_{x(0)}d(u(0,\cdot))(0)=\mathrm{id}_{N_{x(0)}\mathcal{F}}$ implies that $v_0-u(0,\cdot)_*\mathrm{pr}^{\mathcal{F}}v_0\in T_{x(0)}\mathcal{F}$. Using statement (b), it follows that

$$\pi_*u(0,\cdot)_*\mathrm{pr}^{\mathcal{F}}v_0=\pi_*v_0. \tag{61}$$

By assumption we have

$$\mathrm{pr}^{\mathcal{F}}v_1=\mathrm{hol}^{\mathcal{F}}_x\mathrm{pr}^{\mathcal{F}}v_0=\mathrm{pr}^{\mathcal{F}}u(1,\cdot)_*\mathrm{pr}^{\mathcal{F}}v_0.$$

Using again statement (b), it follows that $\pi_*v_1=\pi_*u(1,\cdot)_*\mathrm{pr}^{\mathcal{F}}v_0$. Combining this with (60) and (61), we obtain $\pi_*v_0=\pi_*v_1$, as claimed. This proves (f) and concludes the proof of Lemma 24. □

The next lemma was used in the proof of Theorem 2. Let M be a C^1-manifold, and X a complete C^1-vector field on M. If g is a Riemannian metric on M then we denote by ℓ and d the induced length funcional and distance function, respectively. Furthermore, for a pair $(t,x_0)\in[0,\infty)\times M$ we write $\ell(t,x_0):=\ell\big([0,t]\ni s\mapsto\varphi^s_X(x_0)\in M\big)$.

26. Lemma (Fast almost periodic orbits)**.** *Let (M,g) be a Riemannian C^2-manifold, X a C^1-vector field on M with compact support, and $f:[0,\infty)\to[0,\infty)$ a continuous function such that $f(0)=0$. Then there*

exists a constant $\varepsilon > 0$ such that for every $(t, x_0) \in [0, \varepsilon] \times M$ satisfying $d\big(x_0, \varphi_X^t(x_0)\big) \leq \ell(t, x_0) f(\ell(t, x_0))$, we have $X(x_0) = 0$.

The proof of this lemma is based on an idea from the proof of Proposition 17, p. 184 in the book [**HZ**]. We need the following.

27. Remark. *If $t \geq 0$, and $x \in W^{1,1}\big([0,t], \mathbb{R}\big)$ is such that $\int_0^t x(s)ds = 0$ then*

$$\|x\|_{L^1([0,t])} \leq t\|\dot{x}\|_{L^1([0,t])}. \tag{62}$$

To see this, note that $\int_0^t x(s)ds = 0$ implies that there is a point $t_0 \in [0,t]$ such that $x(t_0) = 0$. It follows that for every $s \in [0,t]$, we have $|x(s)| = \big|\int_{t_0}^s \dot{x}(s)ds\big| \leq \int_0^t |\dot{x}(s)|ds$. Inequality (62) is a consequence of this.

Proof of Lemma 26. Let M, g, X and f be as in the hypothesis. We denote by $K \subseteq M$ the support of X, by n the dimension of M, for a vector $v \in TM$ we denote by $|v|$ its norm with respect to g, and for $v \in \mathbb{R}^n$ we define $|v|_1 := \sum_{i=1}^n |v^i|$. We choose a finite set $\mathcal{A}$ of surjective C^2-charts $\varphi : U \subseteq M \to \mathbb{R}^n$, such that $K \subseteq \bigcup_{(U,\varphi)\in\mathcal{A}} \varphi^{-1}(B_1)$. Furthermore, we choose a constant C such that

$$|x - y|_1 \leq Cd\big(\varphi^{-1}(x), \varphi^{-1}(y)\big), \quad \big|d(\varphi^{-1})(x)v\big| \leq C|v|_1, \tag{63}$$

for every $(U, \varphi) \in \mathcal{A}$, $x, y \in \bar{B}_2$, and $v \in \mathbb{R}^n$. For $(U, \varphi) \in \mathcal{A}$ we define ε_φ to be the distance between $\varphi^{-1}(\bar{B}_1)$ and $M \setminus \varphi^{-1}(B_2)$, and we define

$$\varepsilon_1 := \min\left\{\frac{\varepsilon_\varphi}{\max_K |X|} \,\middle|\, (U,\varphi) \in \mathcal{A}\right\}. \tag{64}$$

For a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ we denote

$$|T|_{\mathrm{op}} := \max\big\{|Tv|_1 \,\big|\, v \in \mathbb{R}^n,\, |v|_1 = 1\big\}.$$

Since by assumption f is continuous and $f(0) = 0$, there exists $\varepsilon_2 > 0$ such that for every $(U, \varphi) \in \mathcal{A}$, and every $a \in \big[0, \varepsilon_2 \max_K |X|\big]$, we have

$$\varepsilon_2 \max_{\bar{B}_2} |d(\varphi_* X)|_{\mathrm{op}} + C^2 f(a) < 1. \tag{65}$$

We define $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. Let $(t, x_0) \in [0, \varepsilon] \times M$ be such that $d\big(x_0, \varphi_X^t(x_0)\big) \leq \ell(t, x_0) f(\ell(t, x_0))$. We define $x : [0,t] \to M$ by $x(s) := \varphi_X^s(x_0)$. We choose a chart $(U, \varphi) \in \mathcal{A}$ such that $x_0 \in \varphi^{-1}(B_1)$. For $s \in [0,t]$, we have

$$d(x_0, x(s)) \leq \ell(x) = \int_0^s |X \circ x(s')|\, ds' \leq \varepsilon_1 \max_K |X| \leq \varepsilon_\varphi.$$

It follows that $x([0,t]) \subseteq \varphi^{-1}(\bar{B}_2)$. Hence we may define

$$y := \varphi \circ x, \quad v := y(0) - y(t), \quad f : [0,1] \to \mathbb{R}^n,\, f(s) := \dot{y}(s) + v/t.$$

The equality $\dot{x} = X \circ x$ implies that f is C^1, and that

$$\dot{f} = \ddot{y} = \big((\varphi_* X) \circ y\big)^{\cdot} = d(\varphi_* X)(y)\dot{y}. \tag{66}$$

Furthermore, $\int_0^t f(s)ds = 0$. Therefore, denoting by $\|f\|_1$ the L^1-norm of f with respect to $|\cdot|_1$, Remark 27 with x replaced by each component of f implies that

$$\|f\|_1 \leq t\|\dot{f}\|_1 \leq t \max_{\bar{B}_2} |d(\varphi_* X)|_{\mathrm{op}} \|\dot{y}\|_1. \tag{67}$$

Here in the second inequality we used (66). By (63) and one of the hypotheses of the lemma, we have

$$|v|_1 \leq Cd(x_0, x(t)) \leq C\ell(t, x_0) f\big(\ell(t, x_0)\big) \leq C^2 \|\dot{y}\|_1 f\big(\ell(t, x_0)\big). \tag{68}$$

Since $\dot{y} = f - v/t$, inequalities (67,68) yield

$$\|\dot{y}\|_1 \leq \|f\|_1 + |v|_1 \leq \Big(\varepsilon_2 \max_{\bar{B}_2} |d(\varphi_* X)|_{\mathrm{op}} + C^2 f\big(\ell(t, x_0)\big)\Big)\|\dot{y}\|_1. \tag{69}$$

Since $\ell(t, x_0) \leq t \max_K |X| \leq \varepsilon_2 \max_K |X|$, inequality (65) holds with $a := \ell(t, x_0)$. Combining this with (69), it follows that $\|\dot{y}\|_1 = 0$. Hence y is constant, and the same holds for $x = \varphi^{-1} \circ y$. This proves Lemma 26. $\square$

The next lemma implies that the Hofer semi-norm given by (14) is well-defined.

28. Lemma. *Let X be a topological space and $f : [0,1] \times X \to \mathbb{R}$ a continuous function. Assume that there exists a sequence of compact subsets $K_\nu \subseteq X$, $\nu \in \mathbb{N}$ such that $\bigcup_\nu K_\nu = X$. Then the map*

$$[0,1] \ni t \mapsto \sup_{x \in X} f(t, x)$$

is Borel measurable.

Proof of Lemma 28. We choose a sequence $K_\nu \subseteq X$, $\nu \in \mathbb{N}$, as in the hypothesis, and we define

$$f_\nu : [0,1] \to \mathbb{R}, \quad f_\nu(t) := \max\big\{f(t, x) \,\big|\, x \in K_\nu\big\}.$$

Then f_ν is continuous, for every ν, and $f(t) = \sup_{\nu \in \mathbb{N}} f_\nu(t)$, for every $t \in [0,1]$. Hence an elementary argument implies that f is Borel measurable. This proves Lemma 28. $\square$

References

[AMR] R. Abraham, J. E. Marsden, T. Ratiu, *Manifolds, tensor analysis, and applications*, 2nd ed., Applied Mathematical Sciences, **75**, Springer, 1988.

[AF] P. Albers, U. Frauenfelder, *Leaf-wise intersections and Rabinowitz Floer homology*, preprint, October 2008, arXiv:0810.3845v1

[An] D. Andrica, *Critical Point Theory and Some Applications*, Cluj University Press, 2005.

[Ar] V. I. Arnol'd, *Sur une propriété topologique des applications globalement canoniques de la mécanique classique*, C. R. Acad. Sci. Paris **261** (1965), 3719–3722.

[ACL] M. Audin, A. Cannas da Silva and E. Lerman, *Symplectic geometry of integrable Hamiltonian systems*, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser, 2003.

[ALP] M. Audin, F. Lalonde and L. Polterovich, *Symplectic rigidity: Lagrangian submanifolds*, **271-321** in *Holomorphic curves in symplectic geometry*, Progr. Math. **117**, Birkhäuser, Basel, 1994.

[Ba] A. Banyaga, *On fixed points of symplectic maps*, Invent. Math. **56** (1980), no. 3, 215-229.

[Bo] W. M. Boothby, *An introduction to differentiable manifolds and Riemannian geometry*, rev. 2nd ed., Academic Press, 2003.

[Ch] Yu. V. Chekanov, *Lagrangian intersections, symplectic energy, and areas of holomorphic curves*, Duke Math. J. **95** (1998), **no. 1**, 213-226.

[Dr] D. Dragnev, *Symplectic rigidity, symplectic fixed points, and global perturbations of Hamiltonian systems*, Comm. Pure Appl. Math. **61** (2008), **no. 3**, 346-370.

[Du] H. Duan, *Morse functions and cohomology of homogeneous spaces*, **105-133** in *Topics in cohomological studies of algebraic varieties*, Trends Math., Birkhäuser, Basel, 2005.

[DV] I. A. Dynnikov and A. P. Veselov, *Integrable Gradient Flows and Morse Theory*, Algebra i Analiz **8** (1996), no. **3**, 78–103, translation in St. Petersburg Math. J. **8** (1997), no. **3**, 429–446

[Eh] C. Ehresmann, *Les connexions infinitésimales dans un espace fibré différentiable*, Colloque de topologie (espaces fibrés), Bruxelles, 1950, 29-55.

[EH] I. Ekeland, H. Hofer, *Two symplectic fixed-point theorems with applications to Hamiltonian dynamics*, J. Math. Pures Appl. (9) **68** (1989), **no. 4**, 467-489 (1990).

[Fl] A. Floer, *Morse theory for Lagrangian intersections*, J. Differential Geom. **28**, **no. 3** (1988), 513-547.

[Fr] U. Frauenfelder, *The Arnold-Givental Conjecture and moment Floer Homology*, Int. Math. Res. Not. **42** (2004), 2179-2269.

[Gin] V. Ginzburg, *Coisotropic intersections*, Duke Math. J. **140** (2007), **no.1**, 111-163.

[GHV] W. Greub, S. Halperin, R. Vanstone, *Connections, curvature, and cohomology, Vol. II: Lie groups, principal bundles, and characteristic classes*, Pure and Applied Mathematics, Vol. **47-II**, Academic Press, 1973.

[GGK] V. Guillemin, V. Ginzburg, and Y. Karshon, *Moment maps, cobordisms, and Hamiltonian group actions*, Mathematical Surveys and Monographs **98**, AMS, Providence, RI, 2002.

[Gü] B. Gürel, *Leaf-wise coisotropic intersections*, in preparation.

[Ho] H. Hofer, *On the topological properties of symplectic maps*, Proc. Roy. Soc. Edinburgh Sect. **A 115** (1990), **no. 1-2**, 25-38.

[HZ] H. Hofer, E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser Advanced Texts: Basel Textbooks, 1994.

[Ke] E. Kerman, *Displacement energy of coisotropic submanifolds and Hofer's geometry*, J. Mod. Dyn. **2** (2008), no. **3**, 471–497.

[KP] St. G. Krantz, H. R. Parks, *The geometry of domains in space*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser, 1999.

[Ma] C.-M. Marle, *Sous-variétés de rang constant d'une variété symplectique*, Astérisque, **107-108**, Soc. Math. France, Paris, 1983.

[MS] D. McDuff and D.A. Salamon, *Introduction to symplectic topology*, 2nd ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, 1998.

[MM] I. Moerdijk, J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge Studies in Advanced Mathematics, 91, Cambridge Univ. Press, 2003.

[Mol] P. Molino, *Riemannian foliations*, Progress in Mathematics, 73, Birkhäuser, 1988.

[Mos] J. Moser, *A fixed point theorem in symplectic geometry*, Acta Math. **141** (1978), no. 1-2, 17-34.

[TT] C.-L. Terng and G. Thorbergsson, *Taut immersions into complete Riemannian manifolds*, **181-228** in *Tight and taut submanifolds*, Math. Sci. Res. Inst. Publ., **32**, Cambridge Univ. Press, 1997.

[Zi1] F. Ziltener, *The Invariant Symplectic Action and Decay for Vortices*, accepted by the J. Symplectic Geom.

[Zi2] F. Ziltener, *A Maslov map for coisotropic submanifolds*, in preparation.